The reverse dimple in space-time covariance models

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Abstract

One class of spatially isotropic models proposed by Gneiting (2002) has been used as a building block to model various complicated non-separable models. Kent et al. (2011) draw out attention on the counterintuitive presence of possible dimple property associated with these covariance models. In this paper, we first attempt to give a simple approach to model potentially negative value stationary spatial-temporal models. Second, we show that in certain circumstances such spatial-temporal models possess a reverse (pointing upward) dimple.

1 Introduction

Let $Z(\boldsymbol{s},t)$ be a real-valued stationary spatial-temporal process defined on $\mathbb{R}^d \times \mathbb{R}$, where $\boldsymbol{s} \in \mathbb{R}^d$ represents the spatial site in d dimension, and $t \in \mathbb{R}$ represents the time. Assume that the second moments for the random variable exist and are finite. Suppose that $C(\boldsymbol{h},u)$ is the corresponding continuous (which we assume everywhere) and symmetric spatial-temporal covariance function on $\mathbb{R}^d \times \mathbb{R}$, where $\boldsymbol{h} \in \mathbb{R}^d$ represents the spatial lag and $u \in \mathbb{R}$ represents the temporal lag. Stationarity is the first assumption to simplify spatial-temporal models. We now consider some more simplifying assumptions that can be adopted when modelling spatial-temporal covariance structure.

Recently several authors have developed approaches to generate positive-value non-separable models for spatial-temporal processes. It is worth Cressie and Huang (1999), Gneiting (2002) and Kolovos et al. (2004). One class of spatially isotropic models proposed by Gneiting (2002), which has been used as a building block to model various recently introduced non-separable spatial-temporal models, takes the following form

$$C(\boldsymbol{h},u) = \frac{\sigma^2}{\psi(u^2)^{\delta}}\varphi\Big(\frac{|\boldsymbol{h}|^2}{\psi(u^2)}\Big),\ (\boldsymbol{h},u) \in \mathbb{R}^d \times \mathbb{R}. \tag{1}$$

where $\varphi(z)$ and $\psi^{'}(z)$, with $\psi(z) > 0,\ z \geq 0$, are completely monotone functions. $\delta \geq d/2$ and $\sigma^2 > 0$ are scalar parameters. Without loss of generality we assume $\varphi(0) = 1$ and $\psi(0) = 1$. Model (1) suggests a strategy to develop new parametric families for valid nonseparable spatial-temporal covariances through closed-form Fourier inversion in $\mathbb{R}^d \times \mathbb{R}$. Using (1) and any completely monotone function $\varphi(z)$, we can construct a large variety of closed-form non-separable fully symmetric covariance functions.

As the spatial or temporal lag increases, we typically expect the covariance to decrease and approaches zero for large distances. But Kent et al. (2011) draw out attention on the counterintuitive presence of possible dimple property associated with the positive covariance surface (1) which contradicts the natural monocity structure of covariance. Covariance can also be negative for some lags exhibiting one or more oscillations before approaching zero. In this paper, we attempt to give a simple approach to model potentially negative value stationary spatial-temporal models. We show that in certain circumstances negative value spatial-temporal models possess a reverse (pointing upward) dimple as well.

2 Negative Value Property

The spatial-temporal covariance function $C_1(\boldsymbol{h}, u)$ has a hole effect in the spatial lag if there exists a real value $\boldsymbol{h}_0^* \geq 0$, where $C_1(\boldsymbol{h}, u)$ takes negative value on $(\boldsymbol{h}_0^*, \infty)$ or exhibiting one or more oscillation before approaching zero in this domain. Similar effect may be happen in the temporal lag. There are at least three approaches to construct hole effect covariance models in purely spatial/temporal setting. A first approach is called product correlations. The second approach is convolution procedure and the third approach is turning bands operator (Matheron, 1973; Gneiting, 1999). The extension of these three approaches to the spatio-temporal case is straightforward with introducing a new concept which we will call it negative value effect.

Say that spatial-temporal covariance function $C_1(\boldsymbol{h}, u)$ has a negative value effect in the temporal lag u if there exists a real value $z_0^* > 0$ such that for fixed $|\boldsymbol{h}|^2 > z_0^*$, $C_1(\boldsymbol{h}, u)$ is negative for $u \in (0, u_0^*)$ for some $u_0^* = u_0^*(|\boldsymbol{h}|^2)$ depending on $|\boldsymbol{h}|^2$ and positive elsewhere. Similar effect may be happen in the temporal lag.

In this paper we will focus on the turning bands operator. The turning bands operator transforms a spatial-temporal function $C(\boldsymbol{h}, u)$, in $\mathbb{R}^d \times \mathbb{R}$, $d \geq 3$ into

$$C_1(\boldsymbol{h}, u) = C(\boldsymbol{h}, u) + (d-2)^{-1}\boldsymbol{h}\frac{\partial}{\partial \boldsymbol{h}}C(\boldsymbol{h}, u) \tag{2}$$

in $\mathbb{R}^{d-2} \times \mathbb{R}$. The operation preserves the local behavior of the covariance function $C(\boldsymbol{h}, u)$ at the origin. The advantage is that, starting from a well-known class of space-time models, with a simple procedure, one can construct covariance functions for spatial-temporal data which can be negative in some part of their domain of definition; if $C(\boldsymbol{h}, u)$ is nonnegative then $C_1(\boldsymbol{h}, u)$ will possess hole effect in spatial lag $\boldsymbol{h}$. More interestingly, in certain circumstances this model will attain negative values in temporal lag u too.

3 Reverse Dimple Property

Applying the turning bands operator to Gneiting model results in

$$C_1(\boldsymbol{h}, u) = C(\boldsymbol{h}, u)\left(1 - 2(d-2)^{-1}Q(z)\right), \ d \geq 3 \tag{3}$$

where $C(\boldsymbol{h}, u)$ is given in (1), $z = |\boldsymbol{h}|^2/\psi(u^2)$ and

$$Q(z) = -z\varphi'(z)/\varphi(z), \quad z > 0. \tag{4}$$

In this section we show that in certain circumstances the proposed spatial-temporal (3), in addition to the dimple and negative value properties, possesses a reverse dimple. First we need to establish the asymptotic behaviors of $Q(z)$.

We introduce now the main assumption on $Q(z)$ that we will use throughout the rest of the paper. We assume the function $Q(\cdot)$ to have the monotonicity properties. For a non-degenerate self-decomposable distribution, the function $Q(z)$ is guaranteed to be monotone increasing in zSteutel and van Harn (2003, Theorem 2.6).

Now we are going to establish the asymptotic behaviors of $Q(z)$. One key property of $Q(\cdot)$ is $\lim_{z\to 0} Q(z) = 0$. Furthermore, Bernstein functions are concave resulting in $0 \le zQ'(z) \le Q(z)$. Then $0 \le \lim_{z\to 0} zQ'(z) \le \lim_{z\to 0} Q(z) = 0$. It follows that $\lim_{z\to 0} zQ'(z) = 0$.

The limiting behaviour of $Q(z)$ and $zQ'(z)$ as $z \to \infty$ can be either finite or infinite. To investigates the limiting behaviour of $Q(z)$ at infinity we assume that $\varphi(z)$ is a regularly varying function at infinity. The following result is due to Lamperti (1958, Theorem 2).

Theorem 3.1. *Let φ be positive and absolutely continuous with first derivative φ'. If*

$$\lim_{z\to\infty} z\frac{\varphi'(z)}{\varphi(z)} = \alpha, \tag{5}$$

Then $\varphi \in R_\alpha$ i.e. $\varphi(z) = z^\alpha \ell(z)$, $\ell \in R_0$. Conversely, if $\varphi(z) = z^\alpha \ell(z)$, $\ell \in R_0$ and φ' is eventually monotone, then (5) holds, and for $\alpha \neq 0$, $(sgn\,\alpha)\varphi'(z) \in R_{\alpha-1}$. If $\alpha = 0$, then $\varphi'(z) = o\left(\ell(z)z^{-1}\right)$ as $z \to \infty$..

Now we develop the following theorem in order to connect the above results to $Q(z)$.

Theorem 3.2. *If φ is a completely monotone function and regularly varying with index $\nu < 0$, then $Q^{(n)}(z) = o\left(Q(z)z^{-n}\right)$ for $\nu \in (-\infty, 0)$. Furthermore, for $\nu = -\infty$, $Q \in R_1$ and $\lim_{z\to\infty} z^n Q^{(n)}(z)/Q(z) = 0$, $n > 1$.*

Proof. Let φ be completely monotone and first consider $\varphi \in R_\nu$ with finite index $\nu < 0$. For $n = 1$, since φ is completely monotone, φ' is monotone and hence by Theorem 3.1, $\lim_{z\to\infty} z\varphi'(z)/\varphi(z) = \nu$. Furthermore, $-\varphi' \in R_{\nu-1}$. Since $-\varphi'$ is also completely monotone function, again by Theorem 3.1, $\lim_{z\to\infty} z(-\varphi''(z)/(-\varphi'(z)) = \nu - 1$. Hence

$$\begin{aligned}\lim_{z\to\infty} zQ'(z)/Q(z) &= \lim_{z\to\infty} \left(1 + z\varphi''(z)/\varphi'(z) - z\varphi'(z)/\varphi(z)\right) \\ &= 1 + (\nu - 1) - \nu = 0.\end{aligned} \tag{6}$$

Therefore $Q \in R_0$. Induction following this argument yields $\lim_{z\to\infty} z^n Q^{(n)}(z)/Q(z) = 0$ which can be interpreted as $Q^{(n)}(z) = o\left(Q(z)z^{-n}\right)$. For $\nu = -\infty$, we consider separately two cases: $n = 1$ and $n > 1$. The reason is that in the latter case one obtains a more general result than in the former. Note that by similar argument, applying Theorem 3.1, using L'Hôpital's rule and the fact that $\lim_{z\to\infty} \varphi^{(n)}(z) = 0$ we have $\lim_{z\to\infty} zQ'(z)/Q(z) = 1$, then $Q \in R_1$ but $\lim_{z\to\infty} z^n Q^{(n)}(z)/Q(z) = 0$, $n > 1$. □

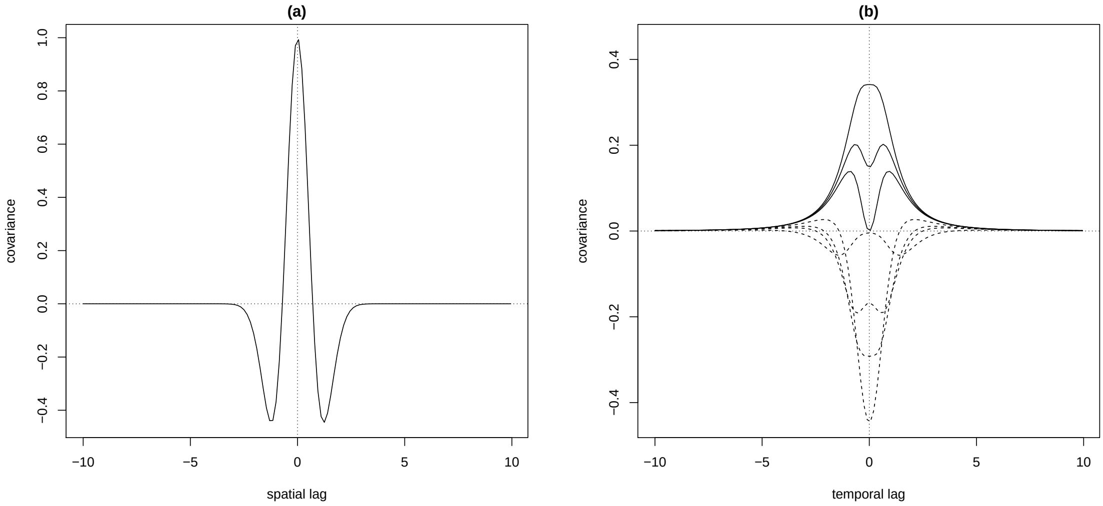

Figure 1: (a): Spatial cross section of space time covariance (8) with parameters $\sigma^2 = 1$ and $\delta = 3/2$, for fixed value $u = 0$. A hole effect is seen in $|\boldsymbol{h}| > 0.707$. (b): Temporal cross sections of space time covariance (8), for fixed values of spatial lag. From top to bottom: $|\boldsymbol{h}| = 0.525, 0.625, 0.707, 2.85, 1.901, 1.651, 1.225$. A dimple is seen for $|\boldsymbol{h}| > 0.525$, a reverse dimple is seen for $|\boldsymbol{h}| > 1.651$ and deepest negative value is seen for $|\boldsymbol{h}| = 1.225$.

The spatial-temporal covariance function $C_1(\boldsymbol{h}, u)$ has a dimple and reverse dimple in the temporal lag u if there exist two real values $0 < z_1^* < z_2^*$ such that the following properties hold.

(a) For fixed $|\boldsymbol{h}|^2 \leq z_1^*$, $C_1(\boldsymbol{h}, u)$ is decreasing in $u \geq 0$.

(b) For fixed $z_1^* < |\boldsymbol{h}|^2 \leq z_2^*$, $C_1(\boldsymbol{h}, u)$ is increasing for $u \in (0, u_1^*)$ for some $u_1^* = u_1^*(|\boldsymbol{h}|^2) > 0$ depending on $|\boldsymbol{h}|^2$, and decreasing for $u \in (u_1^*, \infty)$(dimple property).

(c) For fixed $|\boldsymbol{h}|^2 > z_2^*$, $C_1(\boldsymbol{h}, u)$ is decreasing for $u \in (0, u_2^*)$ for some $0 < u_2^* = u_2^*(|\boldsymbol{h}|^2) < u_1^*$ depending on $|\boldsymbol{h}|^2$, increasing for $u \in (u_2^*, u_1^*)$ and decreasing for $u \in (u_1^*, \infty)$(reverse dimple property).

The following theorem investigates the circumstances under which the proposed spatial-temporal (3), will possess a reverse dimple.

Theorem 3.3. *Consider spatial-temporal model (3) with $Q(z)$ defined by (4). Assume φ is regularly varying with index $-\nu$, $0 < \nu \leq \infty$ and Q is increasing in $z > 0$, then*

(a) if $\lim_{z\to\infty} Q(z) \geq (d-2)/2$, there exist "hole effects " on spatial lag and "negative values" on temporal lag.

(b) if $(d-2)/2 < \lim_{z\to\infty} Q(z) \leq \delta$, then spatial-temporal model (3) has a dimple in u.

(c) if $\delta < \lim_{z\to\infty} Q(z) \leq \infty$, then spatial-temporal model (3) has a dimple and a reverse dimple in u.

Proof. (a) Since $C(\boldsymbol{h}, u)$ is always positive, then (3) attains negative value in spatial and temporal lags if $Q(z) > (d-2)/2$. Q is increasing in $z > 0$ so it is possible to define $z_0^* = Q^{-1}((d-2)/2)$. Hence for each $u \geq 0$ there exists a spatial lag $|\boldsymbol{h}_0^*| = \{z_0^* \psi(u^2)\}^{1/2}$, where $C_1(\boldsymbol{h}, u)$ is negative on $(|\boldsymbol{h}_0^*|, \infty)$ which is precisely the condition of presence of "hole effect " on spatial lag. For each real-valued $\boldsymbol{h}$ satisfying $\psi^{-1}(|\boldsymbol{h}|^2/z_0^*) \geq 0$, $C_1(\boldsymbol{h}, u)$ is always "negative value" on temporal lag $(0, u_0^*)$ where $u_0^* = \{\psi^{-1}(|\boldsymbol{h}|^2/z_0^*)\}^{1/2}$.

(b) Taking partial derivative of (3) with respect to u

$$\begin{aligned}\frac{\partial}{\partial u} C_1(\boldsymbol{h}, u) &= \frac{4\sigma^2 u \psi'(u^2)}{(d-2)\psi(u^2)^{\delta+1}} \varphi(z) \left\{ zQ'(z) - (Q(z) - \delta) \left(Q(z) - \frac{d-2}{2} \right) \right\} \\ &= \frac{4\sigma^2 u \psi'(u^2)}{(d-2)\psi(u^2)^{\delta+1}} \varphi(z) P(z), \text{ say,} \end{aligned} \tag{7}$$

where $P(z) = zQ'(z) - (Q(z) - \delta)\left(Q(z) - \frac{d-2}{2}\right)$. The assumptions on φ, ψ and d in spatial-temporal model (3) ensure that the factor $4u\sigma^2(d-2)^{-1}\varphi(z)\psi'(u^2)/\psi(u^2)^{\delta+1}$ is always non-negative for $u > 0$ in (7) taking zero only as $z \to \infty$. Hence it is sufficient to investigate roots of $P(z)$ in order to discuss the behaviour of the covariance function. First note that $P(0) = \lim_{z \to 0} P(z) = -\frac{d-2}{2}\delta < 0$. Note also that by assumption Q is increasing in $z > 0$, so $zQ'(z) > 0$. Hence $P(z)$ is always positive provided that $\frac{d-2}{2} < Q(z) < \delta$ suggesting a change in sign of $P(z)$ from negative to positive in $[0, z_0^*]$. Furthermore, $-(Q(z) - \delta)(Q(z) - \frac{d-2}{2})$ will monotonically increase in region $z \in [0, z_0^*]$. Hence there exists a unique $0 < z_1^* < z_0^*$ where $P(z_1^*) = 0$.

(c) $P(z)$ is decreasing monotonically on region $[Q^{-1}(\delta/2 + d/4), \infty]$. Suppose φ is rapidly varying, i.e. $\varphi \in R_{-\infty}$, then $Q(\infty)$ is infinite and applying Theorem 3.2, $P(\infty) = -\infty$ always. Hence by the mean value theorem, there always exists a unique point $Q^{-1}(\delta/2 + d/4) < z_2^* < \infty$ where $P(z_2^*) = 0$.

On the other hand, suppose φ is regularly varying with finite index $-\nu$. Then $Q(\infty)$ is finite, i.e. $Q(\infty) = \nu$, $0 < \nu < \infty$ and by Theorem 3.2,

$$\lim_{z \to \infty} P(z) = -(\nu - \frac{d-2}{2})(\nu - \delta).$$

Hence $P(z)$ will pass zero as $z \to \infty$ if $\lim_{z \to \infty} Q(z) = \nu > \delta$.

□

For example starting from space-time model (1) in $\mathbb{R}^3 \times \mathbb{R}$, applying $\varphi(z) = \exp(-z)$ and $\psi(z) = (z+1)$ in (2), with turning band procedure, one can construct the following covariance function for spatial-temporal data in $\mathbb{R} \times \mathbb{R}$

$$C_1(\boldsymbol{h}, u) = \left(1 - \frac{2|\boldsymbol{h}|^2}{1+u^2}\right) C(\boldsymbol{h}, u), \tag{8}$$

where

$$C(\boldsymbol{h}, u) = \frac{\sigma^2}{(1+u^2)^\delta} \exp\left\{\frac{-|\boldsymbol{h}|^2}{1+u^2}\right\}, \ \delta \geq 3/2. \tag{9}$$

For each $u \geq 0$ there exists a spatial lag $|\boldsymbol{h}_0^*| = \sqrt{(1+u^2)/2}$, where $C_1(\boldsymbol{h}, u)$ is negative on $(|\boldsymbol{h}_0^*|, \infty)$ suggesting the presence of "hole effect " on spatial lag. Assuming $\delta = 3/2$, the spatial hole effect for (8) is illustrated in Figure 1(a). The spatial-temporal covariance (9) is always positive while for each $|\boldsymbol{h}| > 0.71$, $C_1(\boldsymbol{h}, u)$ is always "negative value" on temporal lag $(0, u_0^*)$, where $u_0^* = \{\psi^{-1}(|\boldsymbol{h}|^2/z_0^*)\}^{1/2} = \{sqrt(2)|\boldsymbol{h}|^2 - 1\}^{1/2}$. For (9) the dimple effect is described in Kent et al. (2011). For a fixed spatial lag $|\boldsymbol{h}| > 1.225$, the spatial-temporal covariance (9) has a dimple in the temporal lag u. For (8) the dimple and reverse dimple effects are illustrated in Figure 1(b). For a fixed spatial lag $|\boldsymbol{h}| > 0.525$, the spatial-temporal covariance (8) has a dimple in the temporal lag u and hence the spatial covariance is not a decreasing function of the temporal lag u as we normally expect. For a fixed spatial lag $|\boldsymbol{h}| > 1.651$, the spatial-temporal covariance (8) has a reverse dimple in the temporal lag u.

The hight of the dimple and the depth of the reverse dimple, which can be determined by (**??**), are 0.341 and -0.292 respectively. The ratio $C_1(\boldsymbol{h}, u)/C_1(\boldsymbol{0}, 0)$ for (8) cannot be less that -0.446 which will occur at $u = 0$ and $|\boldsymbol{h}| = 1.225$ where the deepest hole effect appears at this site in the spatial lag.

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