A Comparison of Risk Return Relationship in the Portfolio Selection Models

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Abstract

In this paper, we calculate four different kinds of means- AM, GM, HM, and GDM- to investigate the risk-return contour using Markowitz risk minimization and Sharpe's angle maximization models. For a given k value (target portfolio return), the rank order of risk or variance-covariance (υ) can change. In the vertical segment of an efficient frontier curve, we observed v(GDM)> v(HM)> v(GM)> v(AM). At higher k values, the rank changes to v(GDM)> v(HM)> v(AM)> v(GM). That is to say, ranking a portfolio using different kinds of means may well give different rankings depending on what k value one is evaluating. It is also shown the harmonic mean should not be used in the case of a small negative growth rate in stock prices.

Keywords: arithmetic mean, geometric mean, golden mean, harmonic mean, Markowitz risk minimization, Sharpe's angle maximization,

I. Introduction

The foundation of modern investment theory is laid upon the quadratic program portfolio selection model developed more than half century ago by Harry Markowitz (1952, 1956 and 1959). The optimization (risk-minimization) process over mean-variance-covariance space can trace out the efficient frontier curve, which provides the solution space for investors. However, an exact solution cannot be found without the knowledge of a risk free rate on a government bond and an investor's attitude toward risk. To this end, Sharpe (1964) formulated and solved the angle-maximization model in which the risk (standard deviation) adjusted portfolio return (net of risk free rate) is maximized. The Sharpe model provides a convex combination of risk free government bonds and a portfolio of stocks selected based on the criterion of risk minimization. Attitude toward risk such as 20% on bond and 80% on stocks will give investor an exact solution without the knowledge of the indifference (isoutility) curve. Yang et al. (2002) proved that Markowitz risk minimization and Sharpe angle-maximization models are mathematically equivalent given some required portfolio returns and risk free bond rate.

Does the choice of mean returns of stocks in the portfolio matter in the selection process? If so, how different are the optimum solution sets? In this note, we first apply the well-known means: arithmetic, geometric and harmonic means to five companies stocks. In addition, we add a golden mean to the simulation for comparison. The organization of the paper is as follows. Next section introduces the Markowitz risk minimization and Sharpe angle maximization models. Section III describes data and four different means. Section IV performs computer simulations via LINGO to trace out corresponding efficient frontier curves. Section V gives a conclusion.

II. Portfolio selection models with different means

Given a set of n selectable stocks, the purpose of the Markowitz portfolio model is to minimize the weighted risk in terms of variance and covariance of *n* stock returns, or

Minimize $$\upsilon = \sum_i x_i^{\,2}\sigma_i^{\,2} + \sum_i \sum_{i \neq j} x_i x_j \sigma_{ij} \tag{1}$$

Subject to $$\sum_i x_i \overline{R_i} \geq k \tag{2}$$

$$\sum_i x_i = 1 \tag{3}$$

$$x_i \geq 0 \tag{4}$$

Where x_i = the weight or proportion of investment in stock i

$\sigma_i^{\,2}$ = variance of returns in stock i

σ_{ij} = covariance of return between stock *i* and j

$\overline{R_i}$ = expected or average rate of return of stock i

k = target portfolio rate of return

Note that rate of return is frequently calculated as $\frac{P_t - P_{t-1}}{P_{t-1}}$ on which mean, variance and covariance are calculated. Optimum weights ($x_1^*, x_2^*, ..., x_n^*$) for $x_i^* \geq 0$ are the framework under which weighted risk υ can be calculated. Along with a set of k values, we have geometric means calculated according to the growth rate formula (next section) and a set of risk-return values on which an efficient frontier curve can be traced.

However, the exact location cannot be determined with a risk-free bond rate R_F . To expand to risk minimization model, Sharpe (1964) proposed an angle-maximizing model with a highest straight line from R_F that is tangent to the efficient frontier derived from the Markowitz model.

Maximize $$\tan\theta = \frac{\sum_i x_i(\overline{R_i} - R_F)}{(\sum_i x_i^{\,2}\sigma_i^{\,2} + \sum_i \sum_j x_i x_j \sigma_{ij})^{1/2}} \tag{5}$$

Subject to $$\sum_i x_i = 1 \tag{6}$$

$$x_i \geq 0 \tag{7}$$

As is shown by Yang et al. (2002), a given R_F corresponds to a k value in equation (2) of the Markowitz model. Furthermore, the denomination of equation (5) is the square root of the objective function of equation (1). Thus, the Markowitz and Sharpe models exhibit reciprocal relations for a given set of $\overline{R_i}$, R_F and k value.

III. Description of data and characteristics of different mean returns

Monthly stock price of 5 companies from September of 2007 to August of 2008 are calculated to obtain 55 (11*5) rates of return. The arithmetic means (AM) and associated variance and covariance of stock returns are reported in Table 1.

Table 1. Means (%) and Variances (%) of Five Stock Returns

	MA	IBM	JNJ	MCD	WMT
AM	**5.4499%**	**0.6093%**	**0.9410%**	**1.7303%**	**3.0283%**
GM	**4.6235%**	**0.4385%**	**0.8756%**	**1.6012%**	**2.9526%**
HM	**-6.9923%**	**-28.3500%**	**3.1584%**	**4.2577%**	**0.1983%**
GDM	**12.2563%**	**1.4511%**	**1.9342%**	**2.5252%**	**5.1867%**
REG	**4.5127%**	**1.3001%**	**0.6101%**	**0.9135%**	**3.012%**
σ^2_{AM}	**1.7700%**	**0.3386%**	**0.1316%**	**0.2573%**	**0.1562%**
σ^2_{GM}	**1.7768%**	**0.3388%**	**0.1317%**	**0.2575%**	**0.1562%**
σ^2_{HM}	**3.3181%**	**8.7250%**	**0.1808%**	**0.3212%**	**0.2362%**
σ^2_{GDM}	**2.2333%**	**0.3456%**	**0.1415%**	**0.2636%**	**0.2027%**
σ^2_{REG}	**1.7790%**	**0.3433%**	**0.1328%**	**0.2642%**	**0.1562%**

AM=arithmetic mean
GM=geometric mean
HM=harmonic mean
GDM=golden mean
REG= regression-based mean
σ^2_{AM}= variance of stock returns using AM as the central location.
MA= Mastercard Incorporated
IBM=International Business Machines Corp.
JNJ= Johnson & Johnson
MCD= McDonald's Corp.
WMT= Wal-Mart Stores Inc.

An examination on five arithmetic means indicates the highest return is by Masters Charge (MA), followed by Wal-Mart (WMT), McDonalds (MCD), Johnson & Johnson (JNJ) and International Business Machine (IBM). As an alternative to arithmetic mean is the often used geometric mean: $(\frac{P_2}{P_1}\cdot\frac{P_3}{P_2}\cdot\frac{P_4}{P_3}\cdots\frac{P_n}{P_{n-1}})^{\frac{1}{n}}-1=(\frac{P_n}{P_1})^{\frac{1}{n}}-1$. Note that when $\frac{P_{i+1}}{P_i}$ exceeds (falls short of) one it implies a position (negative) growth rate as is measured by $\frac{P_{i+1}}{P_i}-1=\frac{P_{i+1}-P_i}{P_i}$. Viewed in this light, an arithmetic mean (AM) is derived additively whereas its geometric mean (GM) is calculated multiplicatively using the same measure of rate of return, $\frac{P_{i+1}}{P_i}-1$ .Well-known in statistics, AM is more sensitive to outliers than is GM and as such GM may be preferred in such cases. During the sample period, the five stock prices underwent substantial changes. For example, the greatest monthly price change was 28.4% while the largest drop registered 13.97% for Masters Charge. From the perspective of risk averseness, AM might not be preferred.

Calculated geometric mean returns follow the same rank order as that of arithmetic mean returns (Table 1).

A harmonic mean may be appropriate for a variable that measures rates of change (e.g., velocity). In business applications, number of shares of stocks of a national fund, a unit of price (e.g., $1,000,000) can purchase could fit into this category. The harmonic mean (HM) is calculated as $\frac{1}{HM} = \frac{\sum_{i=1}^{n} \frac{1}{x_i}}{n}$ where x_i represents rate of return for stock i. For *n=2* and x_i >0, the geometric mean is the square root of arithmetic and harmonic means or $GM^2 = AM \cdot HM$. For a set of clustered numbers, the three means produce very similar values. However, *HM* can be very biased toward a negative value in the presence of a small negative return, i.e., x_i =-0.02 implies $\frac{1}{x_i}$ =-50 which will dominate other positive regular returns. This is the case we encounter in calculating HM for IBM and MA.(e.g., $x_{11} = -0.66\%$ for MA and $x_4 = -0.91\%$ for IBM). For comparison, we report the HM return in Table 1 as well.

The golden mean, also known as golden section and golden ratio from Leonardo da Vinci is a special case of the geometric mean on line segment |AC|=1: a location B between points A and C such as $\frac{|AC|}{|AB|} = \frac{|AB|}{|BC|}$ which leads to the solution $\frac{-1+\sqrt{5}}{2} = 0.618034$. In another word, it is the range ($x_{\max} - x_{\min}$) that determines the value of golden mean (GDM) since it is hinged calculated as $x_{\min} + 0.618034 * (x_{\max} - x_{\min})$. For instance, the GDM return for MA is $-13.97\% + 0.618034 * (28.47\% - (13.97\%)) = 12.256\%$. When the range plays a key role in determining a representative value, GDM may be a viable candidate and are reported in Table 1. The corresponding variances and covariances of stock returns using the AM, GM, HM, and GDM are shown in Table 2.

Table 2. Variance and Covariance Using Five Means

IV. A comparison of simulation results

An efficient frontier generally consists of two parts: a vertical section and a concave part. The vertical part indicates equation (2) holds with strict inequality (>*k*). That is the portfolio return at optimality exceeds the minimum required rate k. For k = 2% (annualized), which is obviously too low. The optimum portfolio return far exceeds k =2% in the case of AM. If we arbitrarily impose equation (2) with an equality sign (k =2%), there will exist no feasible solution, for the lowest annual average rate of return is 7.317% (IBM), and as such the vertical sections of the efficient frontier curve starts to bend at $k \approx$ 20%, k =19%, k =18%, and k =31% in the cases of AM, GM, HM and GDM respectively (Table 3). A perusal of Table 3 indicate that for range of k <18%, the portfolio risks in the Markowitz model manifest the following rank order 0.1225 (AM)< 0.1226(GM)< 0.1421(HM)< 0.1481(GDM) for each given k value.

As far as minimum risk level where efficient frontier is vertical is concerned, the risk levels of GM and HM are bounded by that AM curve and GDM. Given that GDM are highest in all four means, its variance-covariance are greatest at lower level of k values. The variances of HM are even greater because of negative mean returns. However, five of ten covariances in HM are negative and thus the portfolio risk could be reduced from proper diversification. The high values of the GDM returns translate into higher variances and covariances when compared to that of AM and GM. It is little wonder that the efficient frontier has the largest risk in the vertical segment or v(GDM)> v(HM)> v(GM)> v(AM). As *k* value increases to 30% the risk become the greatest for HM that has 5 largest variances of all the four means. The rank of the portfolio risk become v(HM)> v(GM) > v(AM)> v(GDM). Note that v(GDM) is still at its vertical segment at k=0.30% (see Table 3).

From $k \geq 0.34$ and on the risk in the case of GM become the greatest because the choice set has dwindled: only WMT (35.436%) and MA (55.477%) have the mean return greater than 34% and as such most of the weights go to WMT and MA regardless of their risk levels. On the contrary risk of GDM assumes its smallest value since most of the five stocks have higher returns: 147%, 22.24%, 30.3%, 23.208% and 17.412% for MA, WMT, MCD, JNJ and IBM respectively. HM has shown smallest risk due to negative covariance. However, it must be pointed out that small negative returns are biased and misleading in the sense that the double reciprocity formula assigns too much weight to small negative return (slight price decrease in stock). The risk associated with AM for $k \geq 34\%$ is slightly less than that of GM since GM is less sensitive to outliers than is AM, and as such AM has a slightly larger choice set (smaller risk) at high k value. The four efficient frontiers are shown in Figure1.

For a given risk free rate R_F, the greatest tangent angle attainable between the ray from R_F and the efficient frontier curve can be calculated from equation (5). By verifying R_F we find the optimal tangent angle via LINGO and report the portfolio returns portfolio risk (υ) and tangent angles (tan θ) in Table4. An inspection of Table 4 reveals that Sharpe's solutions with GDM dominates that with HM for R_F =3%, 4%, 5% and 6% because return under GDM are greater but risks smaller than that under HM. For R_F =7% and 10%, returns and risks are both greater under GDM in comparison to HM. Hence we calculate Sharpe's index or its $\tan\theta$ and used as the selection citation.

A comparison between AM and GM indicate that risk-return combinations using AM dominate that using GM: higher return with less risk from R_F =3% through 7%. At R_F $=10\%$, both the portfolio return and risk are greater using AM: 0.3612>0.3465 and 0.2184>0.2104. As a consequence, we compare tan θ and find that under AM is greater 0.5589>0.5373.

Table 3. Risk-Return Combination Using Five Means
Table 4. Angle Maximization Solutions Using Five Means

$Rp = \sum x_i \bar{R}_i =$ Portfolio return
$v =$ Variance and covariance of 5 stocks returns
$tan\theta = \sum x_i(\bar{R} - R_F)/\sqrt{v} =$ Portfolio premium net of risk free rate per risk

V. Concluding remarks

In this paper, we calculate four different kinds of means—AM, GM, HM, and GDM—to investigate the risk-return contour using Markowitz risk minimization and Sharpe's angle-maximization models. For a given k value (target portfolio return), the rank order of risk or variance-covariance (υ) can change. In the vertical segment of an efficient frontier curve, we observed v(GDM)> v(HM)> v(GM)> v(AM). At higher k values, the rank changes to v(GDM)> v(HM)> v(AM)> v(GM). That is to say, ranking a portfolio using different kinds of means may well give different rankings depending on what k value one is evaluating.

When a risk free-rate is added to the Markowitz model, we arrive at the angle-maximization solution. It seems that risk-return combinations under GDM dominate those under HM for the former has high returns and less risk. The same can be said of AM and GM. The combination using AM seems to have greater portfolio returns but less risk. When these exist a trade-off, i.e., higher risk coupled with higher return, Sharpe's index (tan θ) favors both GDM and AM over HM and GM respectively for greater angle translates into high portfolio return(net of R_f) per risk. Care must be exercised though, the results from this paper are limited to the stocks we take in the sample. However, comparative evaluations are needed for a comprehensive analysis on a portfolio performance. In sum, GM is less sensitive to outliers and hence is more suitable for conservative strategy. GDM is determined by the size of its range and tends to offer an optimistic forecast on mean return especially when there exist a few unusually large positive returns. Finally, HM is not appropriate if some of the returns (%) are small and negative. In that case, HM ought to be removed from the analysis.

Figure 1. Efficient Frontier Curves Using Five Different Means

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