

A General Bias Correction Method for the Estimation of Weibull Common Shape Parameter

Yan Shen^{1,2}

¹ Department of Statistics, Xiamen University, Xiamen, Fujian, P.R.CHINA

² Corresponding author: Yan Shen, e-mail: sheny@xmu.edu.cn

Abstracts

The estimation of the common shape parameter across different Weibull populations is an important and general problem in reliability analysis. However, it is widely recognized that the estimation accuracy will be largely affected by small sample size, and many involved Weibull groups, etc. This paper aims to study a general method for correcting the biases rooted in least squares estimator and maximum likelihood estimator of the common shape parameter of Weibull populations. Simulation results show the effectiveness and robustness of the method in bias correction for almost all combinations of sample size and number of populations involved. The method can be extended to more complicated failure data.

Keywords: Stochastic expansion, least squares estimator, maximum likelihood estimator, bootstrap, Monte Carlo simulation

1. Introduction

Weibull distribution is a parametric model popular and widely used in reliability engineering, especially for the analysis of failure time data. Suppose the failure time of an item, denoted by T , follows a Weibull distribution $WB(\alpha, \beta)$. Then the probability density function (pdf) of T has the form:

$$f(t) = \alpha^{-\beta} \beta t^{\beta-1} \exp\{-(t/\alpha)^\beta\}, t \geq 0,$$

where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter. As known, it is the shape parameter β that primarily determines the shape of Weibull distribution. Therefore, to estimate β accurately is of great importance when using Weibull distribution as a candidate model for fitting failure time data.

Two most common methods for Weibull shape parameter estimation are the maximum likelihood method and the least squares method. However, both maximum likelihood estimator (MLE) and least squares estimator (LSE) can be rather biased, in particular for small sample size, many Weibull populations involved or heavily censored data, etc. To address this problem, different methods have been proposed to correct biases rooted in the estimators (Yang and Xie, 2003; Zhang et al., 2006). A more general problem is the estimation of the common shape parameter across different Weibull populations, whose importance was earlier emphasized by Lawless (1982, p.178); a calibrated maximum likelihood estimator was discussed in a recent work of Shen and Yang (2013). In this paper, we also focus on the bias-correction of Weibull common shape estimation, and attempt to continue the work of Shen and Yang (2013) by extending the method to LSE and comparing the performances of the bias-corrected MLE and LSE.

The method is based on a third-order stochastic expansion for parameter estimators (Rilstone et al., 1996) and a simple bootstrap procedure for estimating various expectations involved in the expansion (Yang, 2012). The method is general and simple, as it is essentially applicable to any available smooth estimating equation for the parameters of interest and theoretically need to calculate only the derivatives of estimating functions. Simulation experiments are designed and carried out to assess the performance of the general bias-correction method for MLE and LSE. The results

show that the method is effect and robust in correcting the biases of the two estimators. With the new general method, the bias correction can be easily extended to more complicated failure data, as well as other parametric models. The paper is organized as follows. Section 2 describes the general methodology. Section 3 presents the bias-corrected MLE and LSE for Weibull populations. Section 4 presents Monte Carlo results. Section 5 concludes the paper.

2. General method

Consider a general class of \sqrt{n} -consistent estimator defined by estimating equation $\hat{\theta}_n = \arg\{\psi_n(\theta) = 0\}$, where $\psi_n(\theta) \equiv \psi_n(\mathbf{X}_n; \theta)$ is $l \times 1$ vector-valued function normalized to be of order $O_p(n^{-1/2})$, $\mathbf{X}_n = (X_1, \dots, X_n)$ are the observed data and θ is a $l \times 1$ parameter vector. In studying the finite sample properties of the estimator $\hat{\theta}_n$, Rilstone et al. (1996) developed a stochastic expansion from which bias-correction on $\hat{\theta}_n$ can be made. Let $H_r(\theta) = \nabla^r \psi_n(\theta)$, $r = 1, 2, 3$, be the matrix of the r th order partial derivative of $\psi_n(\theta)$. Under some smoothness requirements on $\psi_n(\theta)$, a stochastic Taylor expansion for $\hat{\theta}_n$ at the true parameter value θ_0 is

$$\hat{\theta}_n - \theta_0 = a_{-1/2} + a_{-1} + a_{-3/2} + o_p(n^{-3/2}) \tag{1}$$

where $a_{-s/2}$ represents terms of order $O_p(n^{-s/2})$ for $s = 1, 2, 3$, and they are

$$\begin{aligned} a_{-1/2} &= \Omega_n \psi_n \\ a_{-1} &= \Omega_n H_{1n}^0 a_{-1/2} + \frac{1}{2} \Omega_n E(H_{2n})(a_{-1/2} \otimes a_{-1/2}) \\ a_{-3/2} &= \Omega_n H_{1n}^0 a_{-1} + \frac{1}{2} \Omega_n H_{2n}^0 (a_{-1/2} \otimes a_{-1/2}) + \frac{1}{2} \Omega_n E(H_{2n})(a_{-1/2} \otimes a_{-1} + a_{-1} \otimes a_{-1/2}) \\ &\quad + \frac{1}{6} \Omega_n E(H_{3n})(a_{-1/2} \otimes a_{-1/2} \otimes a_{-1/2}) \end{aligned}$$

where \otimes represents the Kronecker product, $\psi_n \equiv \psi_n(\theta_0)$, $H_r \equiv H_r(\theta_0)$, $H_r^0 = H_r - E(H_r)$, $r = 1, 2, 3$, and $\Omega_n = -[E(H_{1n})]^{-1}$.

Taking expectation on (1) gives a second-order bias for $\hat{\theta}_n$, $B_2(\hat{\theta}_n) = E(a_{-1/2} + a_{-1})$ and a third-order bias $B_3(\hat{\theta}_n) = E(a_{-1/2} + a_{-1} + a_{-3/2})$. If there is only a sole parameter interested, say θ , then based on Yang (2012) the 2nd- and 3rd-order biases for the estimator $\hat{\theta}_n$ are reduced to

$$B_2(\hat{\theta}_n) = 2\Omega_n E(\psi_n) + \Omega_n^2 E(H_{1n} \psi_n) + \frac{1}{2} \Omega_n^3 E(H_{2n}) E(\psi_n^2) + O_p(n^{-3/2}) \tag{2}$$

$$\begin{aligned} B_3(\hat{\theta}_n) &= 3\Omega_n E(\psi_n) + 3\Omega_n^2 E(H_{1n} \psi_n) + \frac{3}{2} \Omega_n^3 E(H_{2n}) E(\psi_n^2) + \Omega_n^3 E(H_{1n}^2 \psi_n) \\ &\quad + \frac{1}{2} \Omega_n^3 E(H_{2n} \psi_n^2) + \frac{3}{2} \Omega_n^4 E(H_{2n}) E(H_{1n} \psi_n^2) + \frac{1}{2} \Omega_n^5 (E H_{2n})^2 E(\psi_n^3) \\ &\quad + \frac{1}{6} \Omega_n^4 E(H_{3n}) E(\psi_n^3) + O_p(n^{-2}). \end{aligned} \tag{3}$$

Using the two biases $B_2(\hat{\theta}_n)$ and $B_3(\hat{\theta}_n)$, we can perform bias corrections on $\hat{\theta}_n$ or $\hat{\theta}_n^*$, provided that analytical expressions for the various expected quantities in the expansion can be derived and estimated.

3. Bias correction for shape parameter

For the i th Weibull population $WB(\alpha_i, \beta)$, ($i = 1, 2, \dots, k$), let t_{ij} ($j = 1, 2, \dots, n_i$)

be the observed failure times of n_i independent and identically distributed items in the sample, and $n = \sum_{i=1}^k n_i$ be the total number of items. The aim here is to estimate the common shape parameter of k Weibull populations with complete observed data. Thus, taking expansion on concentrated estimating function is a much more effective and simpler way.

3.1. For LSE

The reliability function of Weibull distribution is $R(t) = 1 - F(t) = \exp[-(t/\alpha)^\beta]$ for $t \geq 0$. Taking algorithm twice on both sides of $R(t)$ yields a linear form $\ln[-\ln R(t)] = \beta \ln t - \beta \ln \alpha$. Let $Y = \ln[-\ln R(t)]$ and $X = \ln t$. Then we have $Y = \beta X - \beta \ln \alpha$. Let $t_{i(1)} < t_{i(2)} < \dots < t_{i(n_i)}$ be the ordered sample data for the i th Weibull population, $i = 1, 2, \dots, k$. The values of X are obtained by $x_{ij} = \ln t_{i(j)}$ and the values of Y can be given by $y_{ij} = \ln[-\ln(1 - F_{ij})]$, where F_{ij} is empirical cumulative distribution function at failure time $t_{i(j)}$ and can be estimated by $F_{ij} = (j - 0.3)/(n + 4)$ for complete data. Least squares method requires minimizing the following function

$$g(\beta, \alpha_1, \dots, \alpha_k) = \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \beta x_{ij} + \beta \ln \alpha_i)^2$$

which is the sum of the squared distances between Y and the estimate of Y . Minimizing $g(\beta, \alpha_1, \dots, \alpha_k)$ with respect to α_i ($i = 1, \dots, k$) gives the constrained LSEs

$$\tilde{\alpha}_{n,i} = \exp(\bar{x}_i - \bar{y}_i / \beta).$$

Representing α_i in terms of β reduces $g(\beta, \alpha_1, \dots, \alpha_k)$ to $\tilde{g}(\beta)$,

$$\tilde{g}(\beta) = \sum_{i=1}^k \sum_{j=1}^{n_i} [y_{ij} - \beta x_{ij} - (\bar{y}_i - \beta \bar{x}_i)]^2 = \sum_{i=1}^k [y_{ij} - \beta x_{ij} + \beta \bar{x}_i - \bar{y}_i]^2$$

Minimizing $\tilde{g}(\beta)$, or equivalently solving $\tilde{\psi}_n(\beta) \equiv n^{-1} \frac{d}{d\beta} \tilde{g}(\beta) = 0$, where

$$\begin{aligned} \tilde{\psi}_n(\beta) &= n^{-1} \frac{d\tilde{g}(\beta)}{d\beta} = -2n^{-1} \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \beta x_{ij} + \beta \bar{x}_i - \bar{y}_i)(x_{ij} - \bar{x}_i) \\ &= -2n^{-1} \left[\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)(x_{ij} - \bar{x}_i) - \beta \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \right] \end{aligned}$$

gives the unconstrained LSE $\tilde{\beta}_n$ and the unconstrained LSEs of α_i as $\tilde{\alpha}_{n,i} \equiv \tilde{\alpha}_{n,i}(\tilde{\beta}_n)$, $i = 1, \dots, k$. After some simple algebra, we have

$$\begin{aligned} H_{1n}(\beta) &= \frac{d}{d\beta} \tilde{\psi}_n(\beta) = 2n^{-1} \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \\ H_{2n}(\beta) &= H_{3n}(\beta) = 0 \end{aligned}$$

As $H_{2n}(\beta)$ and $H_{3n}(\beta)$ are zero valued, the bias terms $B_2(\tilde{\beta}_n)$ and $B_3(\tilde{\beta}_n)$ are simplified to

$$\begin{aligned} B_2(\tilde{\beta}_n) &= 2\Omega_n E(\tilde{\psi}_n) + \Omega_n^2 E(H_{1n} \tilde{\psi}_n) + O_p(n^{-3/2}), \\ B_3(\tilde{\beta}_n) &= 3\Omega_n E(\tilde{\psi}_n) + 3\Omega_n^2 E(H_{1n} \tilde{\psi}_n) + \Omega_n^3 E(H_{1n}^2 \tilde{\psi}_n) + O_p(n^{-2}). \end{aligned}$$

where $\Omega_n = -[E(H_{1n})]^{-1}$. The above two equations then lead immediately to the two bias-corrected LSEs of β ,

$$\tilde{\beta}_n^{bc2} = \tilde{\beta}_n - \hat{B}_2(\tilde{\beta}_n) \quad \text{and} \quad \tilde{\beta}_n^{bc3} = \tilde{\beta}_n - \hat{B}_3(\tilde{\beta}_n).$$

where the sign $\hat{}$ implies that the quantity is an estimate and the way that the

estimates are obtained will be discussed in subsection 3.3.

3.2. For MLE

Following a similar deduction as above, the constrained MLEs of α_i are

$$\hat{\alpha}_{n,i}(\beta) = \left(\frac{1}{n_i} \sum_{j=1}^{n_i} t_{ij}^\beta \right)^{1/\beta}$$

and the concentrated log-likelihood function is

$$LL_n^c(\beta) = \sum_{i=1}^k n_i \log n_i - n + n \log \beta + (\beta - 1) \sum_{i=1}^k \sum_{j=1}^{n_i} \log t_{ij} - \sum_{i=1}^k n_i \log \sum_{j=1}^{n_i} t_{ij}^\beta.$$

Maximizing $LL_n^c(\beta)$, or equivalently solving $\tilde{\psi}_n(\beta) \equiv n^{-1} \frac{d}{d\beta} LL_n^c(\beta) = 0$, where

$$\tilde{\psi}_n(\beta) = \frac{1}{\beta} + \frac{1}{n} \sum_{i=1}^k \sum_{j=1}^{n_i} \log t_{ij} - \frac{1}{n} \sum_{i=1}^k \left(\frac{\sum_{j=1}^{n_i} t_{ij}^\beta \log t_{ij}}{\sum_{j=1}^{n_i} t_{ij}^\beta} \right)$$

gives the unconstrained MLE $\hat{\beta}_n$ as well as $\hat{\alpha}_{n,i} \equiv \hat{\alpha}_{n,i}(\hat{\beta}_n)$, $i=1, \dots, k$. Let H_{rn} now be the r th derivative of $\tilde{\psi}_n(\beta)$, $r=1, 2, 3$, then the expressions of H_{rn} ($r=1, 2, 3$) are, respectively,

$$\begin{aligned} H_{1n}(\beta) &= \frac{d}{d\beta} \tilde{\psi}_n(\beta) = \sum_{i=1}^k \frac{n_i}{n} \left(-\frac{1}{\beta^2} - \frac{\Lambda_{2,i}}{T_{\beta,i}} + \frac{\Lambda_{1,i}^2}{T_{\beta,i}^2} \right) \\ H_{2n}(\beta) &= \frac{d^2}{d\beta^2} \tilde{\psi}_n(\beta) = \sum_{i=1}^k \frac{n_i}{n} \left(\frac{2}{\beta^3} - \frac{\Lambda_{3,i}}{T_{\beta,i}} + \frac{3\Lambda_{1,i}\Lambda_{2,i}}{T_{\beta,i}^2} - \frac{2\Lambda_{1,i}^3}{T_{\beta,i}^3} \right) \\ H_{3n}(\beta) &= \frac{d^3}{d\beta^3} \tilde{\psi}_n(\beta) = \sum_{i=1}^k \frac{n_i}{n} \left(-\frac{6}{\beta^4} - \frac{\Lambda_{4,i}}{T_{\beta,i}} + \frac{4\Lambda_{1,i}\Lambda_{3,i}}{T_{\beta,i}^2} + \frac{3\Lambda_{2,i}^2}{T_{\beta,i}^2} - \frac{12\Lambda_{1,i}^2\Lambda_{2,i}}{T_{\beta,i}^3} + \frac{6\Lambda_{1,i}^4}{T_{\beta,i}^4} \right) \end{aligned}$$

where $T_i \equiv T_i(\beta) = \sum_{j=1}^{n_i} t_{ij}^\beta$, $\Lambda_{si} \equiv \Lambda_{si}(\beta) = \sum_{j=1}^{n_i} t_{ij}^\beta (\log t_{ij})^s$, $s=1, 2, 3, 4$, $i=1, \dots, k$.

Replacing the corresponding quantities in (2)-(3) yields the two bias-corrected MLEs, $\hat{\beta}_n^{bc2} = \hat{\beta}_n - \hat{B}_2(\hat{\beta}_n)$ and $\hat{\beta}_n^{bc3} = \hat{\beta}_n - \hat{B}_3(\hat{\beta}_n)$. The details of this subsection can refer to Shen and Yang (2013).

3.3. A bootstrap method for practical implement

A feasible approach for calculating the various expectations in the bias formula is the bootstrap method (Yang, 2012). In this work, the parametric bootstrapping is suggested, as the distribution of the model is completely specified. A great advantage of the parametric bootstrap method is that for either complete or incomplete data, the resampling can be done by directly generating random samples from fitted distribution. The bootstrapping procedure for the LSE is summarized as follows:

Step 1: Compute the LSEs $\tilde{\beta}_n$ and $\tilde{\alpha}_{n,i}$ ($i=1, \dots, k$);

Step 2: For each Weibull population WB($\tilde{\alpha}_{n,i}, \tilde{\beta}_n$), $i=1, \dots, k$, generate n_i random data as a new complete sample, and denote the sample by $(t_{ij}^b, j=1, \dots, n_i)$;

Step 3: Based on t_{ij}^b , $j=1, \dots, n_i$, $i=1, \dots, k$, compute the corresponding quantities

$$\tilde{\psi}_{n,b}(\tilde{\beta}_n), H_{1n,b}(\tilde{\beta}_n), H_{2n,b}(\tilde{\beta}_n) \text{ and } H_{3n,b}(\tilde{\beta}_n);$$

Step 4: Repeat steps (2)-(3) B ($b=1, \dots, B$) times to get sequences of bootstrapped values for $\tilde{\psi}_n(\tilde{\beta}_n)$, $H_{1n}(\tilde{\beta}_n)$, $H_{2n}(\tilde{\beta}_n)$ and $H_{3n}(\tilde{\beta}_n)$.

The bootstrap estimates of various expectations can be obtained as follows. For example, the bootstrap estimates for $E(\tilde{\psi}_n^2)$ and $E(H_{1n}\tilde{\psi}_n)$ are, respectively,

$$\hat{E}(\tilde{\psi}_n^2) = \frac{1}{B} \sum_{b=1}^B [\tilde{\psi}_{n,b}(\tilde{\beta}_n)]^2 \quad \text{and} \quad \hat{E}(H_{1n}\tilde{\psi}_n) = \frac{1}{B} \sum_{b=1}^B H_{1n,b}(\tilde{\beta}_n)\tilde{\psi}_{n,b}(\tilde{\beta}_n).$$

The other expectations in the bias can be similarly bootstrapped.

4. Simulations

Monte Carlo simulations were carried out to investigate the finite sample performances of the proposed method in correcting biases of the LSE and MLE of Weibull common shape parameter. Due to space limitation, only complete sample case is considered in this paper; the numbers of groups are chosen to be $k = 1, 2, 8$ respectively. Tables 1 and 2 summarize the empirical mean, root-mean-squared-error (rmse) and standard error (se) of the original and bias-corrected LSEs or MLEs under different combinations of the values of n, β and k (The results for $k = 8$ not presented here). In simulation experiments, the parametric bootstrapping procedure is adopted, which (i) fits original data to Weibull model, (ii) draws random samples from the fitted distribution with the size being the same as the original sample size. For all the experiments, 5000 replications are run in each simulation and the number of bootstrap is set to be 699.

From the tables, we see that the second-order and third-order bias-corrected MLEs, $\hat{\beta}_n^{bc2}$ and $\hat{\beta}_n^{bc3}$, are generally nearly unbiased and are much superior to the original MLE $\hat{\beta}_n$ regardless of the values of n, β and k . For the LSE, the bias-corrected estimators improve the estimation accuracy in almost all cases, except $k = 1, n = 10, 20$. And the improvement becomes more significant with more Weibull populations involved. Moreover, a phenomenon worthy of noting is that the second-order bias-correction seems sufficient and a higher order bias correction may not be always necessary. This may be due to the numerical instability in the third-order correction term especially when sample size is small.

5. Conclusions

In this paper, we study a general method for correcting the biases of the LSE and MLE of the common Weibull shape parameter. The method is based a third-order stochastic

Table 1. Empirical mean[rmse](se) of MLE- and LSE- type estimators of β , complete data, $k=1$.

10	0.5	0.487[.164](.163)	0.441[.160](.148)	0.441[.395](.148)	0.589[.199](.178)	0.505.153	0.505.153
	0.8	0.776[.258](.256)	0.702[.253](.233)	0.702[.616](.233)	0.936[.311](.279)	0.802.240	0.802.240
	1.0	0.972[.322](.321)	0.880[.316](.293)	0.880[.776](.293)	1.173[.389](.348)	1.005.298	1.005.299
	2.0	1.933[.641](.638)	1.748[.633](.581)	1.748[1.53](.581)	2.341[.781](.702)	2.006.602	2.006[.602](.603)
	5.0	4.831[1.61](1.06)	4.377[1.59](1.46)	4.377[3.92](1.46)	5.833[1.93](1.74)	4.9981.49	4.9991.50
20	0.5	0.478[.109](.107)	0.477[.109](.107)	0.477[.136](.107)	0.537[.110](.103)	0.499.096	0.500.096
	0.8	0.768[.179](.176)	0.766[.179](.176)	0.766[.224](.176)	0.860[.175](.165)	0.800.153	0.801.153
	1.0	0.962[.224](.221)	0.959[.224](.220)	0.959[.283](.220)	1.077[.220](.206)	1.002.192	1.003.192
	2.0	1.915[.438](.430)	1.909[.440](.430)	1.909[.548](.430)	2.151[.434](.407)	2.000.379	2.002.379
	5.0	4.807[1.15](1.13)	4.792[1.15](1.13)	4.792[1.43](1.13)	5.395[1.12](1.05)	5.015.976	5.021.977
50	0.5	0.484[.072](.071)	0.495.072	0.495[.074](.072)	0.515[.060](.058)	0.500.056	0.500.056
	0.8	0.776[.116](.114)	0.794[.117](.116)	0.794[.120](.116)	0.826[.097](.094)	0.803.091	0.803.091
	1.0	0.967[.145](.141)	0.989[.145](.144)	0.989[.148](.144)	1.027[.119](.115)	0.999.112	0.999.112
	2.0	1.940[.290](.284)	1.983.291	1.983[.300](.291)	2.061[.242](.234)	2.004.227	2.005.227
	5.0	4.853[.734](.719)	4.960[.736](.735)	4.960[.757](.735)	5.165[.632](.610)	5.022[.594](.593)	5.023[.594](.593)
100	0.5	0.488[.052](.051)	0.498.052	0.498[.053](.052)	0.507.041	0.500.040	0.500.040
	0.8	0.783[.083](.081)	0.799.083	0.799.083	0.813[.066](.065)	0.802.064	0.802.064
	1.0	0.976[.104](.101)	0.996.103	0.996[.104](.103)	1.014[.080](.079)	1.000.078	1.000.078
	2.0	1.949[.211](.205)	1.989.210	1.989[.211](.210)	2.026[.163](.161)	1.998.159	1.998.159
	5.0	4.885[.532](.520)	4.983.531	4.983[.534](.531)	5.071[.410](.404)	5.001.398	5.001.398

Table 2. Empirical mean[rmse](se) of MLE- and LSE- type estimators of β , complete data, $k=2$.

10	0.5	0.443[.120](.106)	0.467[.116](.112)	0.467[.142](.112)	0.559[.124](.109)	0.500.098	0.498.097
	0.8	0.703[.194](.168)	0.742[.187](.177)	0.742[.222](.177)	0.889[.197](.176)	0.794.157	0.791[.157](.156)
	1.0	0.890[.241](.215)	0.938[.235](.227)	0.938[.288](.227)	1.119[.253](.223)	1.000.199	0.996.198
	2.0	1.771[.485](.427)	1.868[.471](.452)	1.868[.574](.452)	2.231[.507](.451)	1.995.404	1.987[.403](.402)
	5.0	4.433[1.20](1.05)	4.674[1.16](1.11)	4.674[1.42](1.11)	5.580[1.26](1.11)	4.989.997	4.969[.994](.993)
20	0.5	0.458[.087](.076)	0.488[.082](.081)	0.488[.085](.081)	0.527[.073](.068)	0.500.064	0.500.064
	0.8	0.732[.140](.122)	0.779[.131](.130)	0.779[.136](.130)	0.843[.117](.109)	0.800.104	0.799.104
	1.0	0.918[.174](.153)	0.978[.165](.163)	0.978[.172](.163)	1.053[.149](.139)	0.999.132	0.999.132
	2.0	1.826[.356](.310)	1.944[.335](.331)	1.944[.347](.331)	2.104[.294](.275)	1.996.261	1.995.261
	5.0	4.580[.878](.772)	4.878[.831](.822)	4.878[.864](.822)	5.273[.750](.699)	5.003.663	5.000.663
50	0.5	0.475[.057](.051)	0.498.053	0.498[.054](.053)	0.511[.042](.040)	0.500.040	0.500.040
	0.8	0.760[.090](.081)	0.796.085	0.796[.086](.085)	0.817[.068](.066)	0.800.064	0.800.064
	1.0	0.949[.112](.100)	0.994.105	0.994.105	1.019[.084](.082)	0.999.080	0.999.080
	2.0	1.897[.226](.201)	1.987[.212](.211)	1.987[.213](.211)	2.042[.169](.164)	2.001.161	2.001.161
	5.0	4.742[.569](.507)	4.966[.532](.531)	4.966[.535](.531)	5.098[.419](.408)	4.996.400	4.995.400
100	0.5	0.483[.040](.036)	0.498.037	0.498.037	0.505.028	0.500.027	0.500.027
	0.8	0.772[.064](.058)	0.797.059	0.797[.060](.059)	0.808[.045](.044)	0.800.044	0.800.044
	1.0	0.968[.079](.072)	0.999.075	0.999.075	1.011[.058](.057)	1.001.056	1.001.056
	2.0	1.934[.161](.148)	1.997.152	1.997[.153](.152)	2.021[.115](.113)	2.001.112	2.001.112
	5.0	4.835[.399](.363)	4.991.375	4.991[.376](.375)	5.053[.287](.282)	5.003.280	5.003.280

expansion and a simple bootstrap procedure that allows easy estimation of various expected quantities involved in the expansions for bias. Monte Carlo simulation experiments are conducted and the results show that generally the discussed method is able to give satisfactory performances in bias correction regardless of sample size and the number of populations involved; only in the situation of small sample size and one Weibull group, the bias-corrected LSEs did not outperform the original one; but with more populations, the bias-corrected LSEs could greatly improve the estimation accuracy.

Although only the case of complete sample is examined in this paper, the method can be extended to the situations when censored data are encountered (Shen and Yang, 2013). Also, we may infer that the method should be a simple and good method that can be applied to other parametric distributions, as long as the concentrated estimating equations for the parameter(s) of interest, and their derivatives (up to third order) can be expressed in analytical forms. Moreover, possible comparisons between the proposed method and various other estimation approaches would be meaningful.

References

- Lawless, J.F. (1982) *Statistical Models and Methods for Lifetime Data*, John Wiley and Sons.
- Rilstone, P., Srivastava, V.K., and Ullah, A. (1996) "The second-order bias and mean squared error of nonlinear estimators," *Journal of Econometrics*, 75, 369-395.
- Shen, Y., and Yang, Z.L. (2013) "Bias-Correction for Weibull Common Shape Estimation," under review.
- Yang, Z.L. (2012) "A general method for third-order bias and variance corrections on a nonlinear estimator," working Paper, Singapore Management University.
- Yang, Z.L., and Xie, M. (2003) "Efficient estimation of the Weibull shape parameter based on modified profile likelihood," *Journal of Statistical Computation and Simulation*, 73, 115-123.
- Zhang, L.F, Xie, M., and Tang, L.C. (2006) "Bias correction for the least squares estimator of Weibull shape parameter with complete and censored data," *Reliability Engineering and System Safety*, 91, 930-939.