Parameters Estimation of CES and Translog Production Functions Based on SAM in a Regional CGE Model Framework

Guido Ferrari[13] and Luca Secondi[2]
[1]Dipartimento di Statistica, Informatica, Applicazioni "G. Parenti", Università degli Studi di Firenze & Renmin University of China, ITALY & CHINA
[2] Department for Innovation in Biological, Agro-food and Forest systems, University of Tuscia, Viterbo, ITALY
[3]Corresponding author: Guido Ferrari, e-mail: ferrari@disia.unifi.it

Abstract
Numerical specification of regional or economy-wide equilibrium analysis for solution of GE models crucially depends on whether calibration and estimation are performed, so to be capable of reproducing the complete benchmark data set as an equilibrium solution of the model. In this paper, we suggest a new entropy based procedure with nested CES and Translog production functions in which the substitution, efficiency, and distribution parameters, and therefore the elasticities of substitution, are estimated on the basis of the data set represented by the SAM.

Keywords: calibration, estimation, entropy, supports, objective function, elasticity of substitution

1. Introduction

Numerical specification of regional or economy-wide equilibrium analysis for solution of applied general equilibrium models (the so-called Computable General Equilibrium (CGE) models), crucially depends on whether calibration and estimation are reasonably performed, so to be capable of reproducing the complete benchmark data set as an equilibrium solution of the model.

Indeed, the unknown parameters describing technology and preferences[1] in these models are determined partly by calibration, or, on a single data point estimation (Jorgenson (1984)), that is, data from a single inter-industry transactions table of a Social Accounting Matrix (SAM), and partly by econometric estimation, i.e. estimates either existing in literature and used as imputed values or purposively performed on cross-sections or time series data.

Calibration is allowed by some assumptions on technology, such as the log-linearity of the production functions employed in modelling it, which implies that relative shares of inputs in the value of output are fixed.

The way to calibration was opened by Johansen (1960), who in the implementation of an applied general equilibrium model and in describing producer behaviour, retained the fixed coefficients assumption of input output analysis in modelling demands for intermediate goods, employed linear logarithmic or Cobb-Douglas production functions in modelling the substitution between capital and labour services and technical change, implying unity of elasticity of substitution, and used econometric methods to estimate constant rates of technical change.

Since the elasticity of substitution was shown not to be approximately unity, Cobb-Douglas production functions needed to be replaced by Constant Elasticity of Substitution (CES) production functions (either single-stage or nested) that do not imply unity and reflect a trade-off that model builders face between complexity and tractability. Lacking log-linearity, the substitution parameter, and hence its

[1] As in this paper we are interested to model production activity of branches, in what follows the concern will be on producer behaviour only.

transformation, the elasticity of substitution – which values critically affect the results obtained - could no longer be calibrated and have been taken in literature through the so called extraneous estimation approach, and imputed.

Imputation, as well as on the other hands, calibration, have been hardly criticized. Nevertheless, subsequent works in applied GE models have substantially preserved Johansen calibration-imputation mixed approach.

Jorgenson (1984) criticized the assumptions on technology employed in this approach, particularly the fixed coefficients one, and suggested econometric estimation only, consisting of generating complete systems of demand functions for inputs in each branch, with quantities of inputs demanded as functions of prices and output. However, predicted values from a separate production system have the potential to violate product balance conditions for some years of historical data.

In this paper, in the furrow drawn by the calibration and estimation approach, and based on the seeds put by the original Jorgenson's idea of SAM based parameter estimation, we suggest a procedure with nested CES and Translog production functions in which the substitution parameter ρ, the efficiency parameter γ and the distribution parameter δ, and therefore the elasticities of substitution, are estimated on the basis of the data set represented by the SAM..

As far as CES and Translog elasticities of substitution are concerned, no attempts has been made so far to estimate them using the information contained in the SAM, except Ferrari and Manca (2008), who obtained Generalized Maximum Entropy (GME) estimates of the above CES elasticities between VA and IC in a Regional Environmentally Extended CGE model framework based on an environmentally extended SAM for the Italian region Sardinia.

CES functions put a restriction on elasticities of substitution of being constant along the whole isoquant. Translog functions allow for variability of Allen partial elasticities of substitution, are more flexible and linear in parameters. Therefore, we use nested Translog production functions as well to estimate Allen partial elasticities of substitution (AES) that in two factor case are equal to direct AES.

To the above purposes, we make resort to the Generalized Cross-Entropy (GCE) approach that allows to effectively treat ill-posed situations like this.

2. GCE Estimation of the parameters of CES and Translog Production Functions Based on a SAM for Tuscany in a Regional CGE Framework

The production process has been nested in two stages: the first one puts the output *y* as a CES function of Value Added (VA) and Intermediate Consumption (IC); the second one gets the VA as a CES function of Capital (K) and Labour (L). The elasticities of substitution one is interested in are those between VA and IC and between K and L.

CES production functions are widely used in CGE structure to model the production process as it implies the estimation of three parameters only. This parsimony with respect to the number of behavioural parameters comes at a cost in terms of flexibility in representing technology.

In the form for factors inputs, the CES production function is

$$y = \gamma\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-1/\rho} \qquad (1)$$

where y is output, K capital, L labour, and $\gamma, \rho,$ and δ are the efficiency, the substitution, and the distribution parameters, respectively. Admissible values of ρ run from -1 to ∞, which allows the elasticity of substitution $\sigma = \frac{1}{1+\rho}$ to range from $+\infty$, straight line isoquant, to 0. Since there is evidence of elasticity of substitution in different branches less than unity, they imply positive values of $\rho = \frac{1-\sigma}{\sigma}$.

The version of the double-nested 2-factor CES production function we use is

$$y_i = \gamma_i \left[\delta_i X_{1i}^{-\rho_i} + (1-\delta_i) X_{2i}^{-\rho_i}\right]^{-1/\rho_i} \exp e_i;\ i = 1,\dots,N \qquad (2)$$

where X_{1i} and X_{2i} are the two factors, e_i is an error term, and N is the number of observations (branches). Its log-linear form is

$$\log y_i = \log \gamma_i - \rho_i^{-1} \log\left[\delta_i X_{1,i}^{-\rho_i} + (1-\delta_i) X_{2,i}^{-\rho_i}\right] + e_i;\ i = 1,\dots,N \qquad (3)$$

with the efficiency, substitution and share parameters γ_i, ρ_i, and δ_i to be estimated, X_{1i} and X_{2i} being, in turn, *VA* and *CI*, and *K* and *L*, respectively.

Flexible forms of production functions represent an important tool to make the results of the computation of a CGE model more meaningful. The two-output/two input Transcendental Logarithmic (Translog) production frontier, along with its one-output/two input special case, as a second order Taylor series expansion are perhaps the most relevant functions that can be used to model production technologies.

This is very attractive in CGE context since it does not impose any pre-specified restriction on the elasticity of the substitution among production factors: while the CES function assumes constant return to scale along the isoquant, the Translog function exhibits variable return to scale.

In our analysis, we use the following Translog production function:

$$logY_i = \beta_0 + \beta_1 logX_{1i} + \beta_2 logX_{2i} + \frac{1}{2}\beta_{11}(logX_{1i})^2 + \beta_{12}(logX_{1i})(logX_{2i}) + \frac{1}{2}\beta_{22}(logX_{2i})^2 + u_i;\quad i = 1,\dots,N \qquad (4)$$

where Y_i is the output, X_{1i} and X_{2i} are the two inputs, VA and CI in the first nest and K and L in the second nest, the coefficients β_0, $[\beta_1, \beta_2]$, and $[\beta_{11}, \beta_{22}, \beta_{12}]$ represent the unknown parameters to be estimated and u_i, the error term.

In dealing with the approximation of the Translog function to the CET-CES production frontier, a suggested economic interpretation of the Translog parameters is that β_0 can be defined as the efficiency parameter, $[\beta_1, \beta_2]$ as the distribution parameters and $[\beta_{11}, \beta_{22}, \beta_{12}]$ as the substitution parameters.

Since Allen partial elasticity of substitution (APES) in the two-input dimensional Translog function coincides with Hicks elasticity of substitution (HES) and equals Allen direct elasticity of substitution (ADES), we estimate HES in this paper. The cross HES/ADES is therefore:

$$h_{ij} = (\beta_{ij} + M_i M_j)/M_i M_j$$

where M_i is the logarithmic marginal product of factor i.

Database for estimation has been represented by the 30 branches, 2002 SAM for Tuscany. It has been transformed by grouping the 30 branches into 7, based on an ad hoc conversion table(available on request). This has led to the reduced 2002 SAM (available on request).

In order to enlarge the information, and double the number of branches in each group, following Golan, Judge, and Robinson (1994), we have obtained the 2009 reduced SAM (available on request): starting from the 2002 SAM, $\mathbf{X}^{02}$, and using the most recent available macroeconomic information on VA by branch, on Household Consumption by commodity, and on indirect taxes and taxes on import in 2009 from the Regional Accounts of the Italian National Statistical Office (ISTAT), we have obtained a Cross-Entropy (CE) 2009 SAM, $\mathbf{X}^{09,}$ for Tuscany.

SAM based GCE estimates of the parameters of CES production function have been obtained through re-parameterization of equation (3) as follows:

$\boldsymbol{\beta}' = [\gamma,\rho,\delta]' = \mathbf{Zp}$, where $\mathbf{Z}$ is the block diagonal ($K \times KM$), K=3, matrix of known support values for $\boldsymbol{\beta}'$

$$\begin{bmatrix} z_{11}\ z_{12}\ \ldots\ldots\ldots\ z_{1M} & \ldots\ldots & 0\ \ 0\ldots\ldots\ldots\ldots0 & \ldots\ldots & 0\ \ 0\ldots\ldots\ldots\ldots0 \\ 0\ \ 0\ldots\ldots\ldots\ldots0 & \ldots\ldots & z_{21}\ z_{22}\ \ldots\ldots\ldots\ z_{2M} & \ldots\ldots & 0\ \ 0\ldots\ldots\ldots\ldots0 \\ 0\ \ 0\ldots\ldots\ldots\ldots0 & \ldots\ldots & 0\ \ 0\ldots\ldots\ldots\ldots0 & \ldots\ldots & z_{31}\ z_{32}\ \ldots\ldots\ldots\ z_{3M} \end{bmatrix}$$

p is the ($KM \times 1$) vector of unknown probabilities and M is the number of support points for each parameter, which is supposed to be constant over the three parameters and put equal 5, out of the $2 \le M \le \infty$ potential outcomes of the discrete random variable, as the choice of five points represents the greatest improvement in precision. Specifically, as for the efficiency, substitution and distribution parameters we put:

$\gamma = \sum_{f=1}^{F} r_f p_f$, $\rho = \sum_{d=1}^{D} s_d q_d$, $\delta = \sum_{g=1}^{G} t_g b_g$;F=D=G=5.

As for the error term, we put **e** = **Vw,** where **V** is the block diagonal ($N \times NL$) matrix of known support values for **e** and **w** is an ($NL \times 1$) vector of unknown probability weights, where L, again equal five, is the number of support values chosen for each error w_i. Hence, $e_i = \sum_{l=1}^{L} v_{il} w_{il}$; ($i=1,\ldots,N$); L=5.

The objective function is then

$$\begin{aligned} Min\ H(\mathbf{p},\mathbf{p}^0,\mathbf{q},\mathbf{q}^0,\mathbf{b},\mathbf{b}^0\mathbf{w},\mathbf{w}^0) &= \sum_{k=1}^{K}\sum_{f=1}^{F} p_{kf} \ln p_{kf} - \sum_{k=1}^{K}\sum_{f=1}^{F} p_{kf} \ln p_{kf}^0 \\ &+ \sum_{k=1}^{K}\sum_{d=1}^{D} q_{kd} \ln q_{kd} - \sum_{k=1}^{K}\sum_{d=1}^{D} q_{kd} \ln q_{kd}^0 + \sum_{k=1}^{K}\sum_{g=1}^{G} b_{kg} \ln b_{kg} - \sum_{k=1}^{K}\sum_{g=1}^{G} b_{kg} \ln b_{kg}^0 \qquad (5) \\ &+ \sum_{i=1}^{N}\sum_{l=1}^{L} w_{il} \ln w_{il} - \sum_{i=1}^{N}\sum_{l=1}^{L} w_{il} \ln w_{il}^0 \end{aligned}$$

subject to

$$\ln y_i = \ln\left(\sum_{f=1}^{F} r_{kf} p_{kf}\right) - \left(\sum_{d=1}^{D} s_{kd} q_{kd}\right)^{-1}$$
$$\ln\left[\left(\sum_{g=1}^{G} t_{kg} b_{kg}\right) X_{1,i}^{-\left(\sum_{d=1}^{D} s_{kd} q_{kd}\right)} + \left(1 - \left(\sum_{d=1}^{D} s_{kd} q_{kd}\right)\right) X_{2,i}^{-\left(\sum_{d=1}^{D} s_{kd} q_{kd}\right)}\right] + \sum_{l=1}^{L} v_{il} w_{il}$$

and normalization:

$$\sum_{k=1}^{K} p_{kf} = \sum_{k=1}^{K} q_{kd} = \sum_{k=1}^{K} b_{kg} = 1;\ (f = d = g = 1,\ldots,5);\ \sum_{i=1}^{N} w_{il};\ (l = 1,\ldots,5).$$

As regards the Translog parameters, similarly to the CES estimates, we firstly re-parameterized all the unknown coefficient in (4) as follows:

$$\beta_0 = \sum_{f=1}^{F} m_f\, c_f;\ \beta_1 = \sum_{f=1}^{F} m_f\, d_f;\ \beta_2 = \sum_{f=1}^{F} m_f\, g_f;\ \beta_{11} = \sum_{f=1}^{F} m_f\, l_f;$$
$$\beta_{12} = \textstyle\sum_{f=1}^{F} m_f\, r_f;\ \beta_{22} = \sum_{f=1}^{F} m_f\, t_f;\ F = 1,2,\ldots,5,$$

where F=1,2,…5 identifies the support points for all the unknown parameter, and **c**, **d**, **g**, **l**, **r**, **t** are *f-dimensional* vectors of probabilities to be estimated. A similar re-parameterization was carried out for the error terms such that: $u_i = \sum_{l=1}^{L} v_{il} w_{il}$; ($i=1,\ldots,N$); L=5 . The objective function is therefore specified as follows:

$$Min\ H(\mathbf{c},\mathbf{c^0},\mathbf{d},\mathbf{d^0},\mathbf{g},\mathbf{g^0}\mathbf{l},\mathbf{l^0},\mathbf{r},\mathbf{r^0},\mathbf{t},\mathbf{t^0},\mathbf{w},\mathbf{w^0}) = \sum_{f=1}^{F} c_f \ln c_f - \sum_{f=1}^{F} c_f \ln c_f^0 +$$
$$+\sum_{f=1}^{F} d_f \ln d_f - \sum_{f=1}^{F} d_f \ln d_f^0 + \sum_{f=1}^{F} g_f \ln g_f - \sum_{f=1}^{F} g_f \ln g_f^0 + \sum_{f=1}^{F} l_f \ln l_f - \sum_{f=1}^{F} l_f \ln l_f^0 +$$
$$+\sum_{f=1}^{F} r_f \ln r_f - \sum_{f=1}^{F} r_f \ln r_f^0 + \sum_{f=1}^{F} t_f \ln t_f - \sum_{f=1}^{F} t_f \ln t_f^0 + \sum_{i=1}^{N}\sum_{l=1}^{L} w_{il} \ln w_{il} - \sum_{i=1}^{N}\sum_{l=1}^{L} w_{il} \ln w_{il}^0 \quad (5)$$

subject to the consistency-data constraints:

$$\log Y_i = \left(\sum_{f=1}^{F} m_f c_f\right) + \left(\sum_{f=1}^{F} m_f d_f\right)\log X_{1i} + \left(\sum_{f=1}^{F} m_f g_f\right)\log X_{1i} + \frac{1}{2}\left(\sum_{f=1}^{F} m_f l_f\right)$$
$$\left(\log X_{1i}\right)^2 + \left(\sum_{f=1}^{F} m_f r_f\right)\log X_{1i}\log X_{2i} + \frac{1}{2}\left(\sum_{f=1}^{F} m_f t_f\right)\left(\log X_{2i}\right)^2 + \sum_{l=1}^{L} v_{il} w_{il}$$

and the usual normalization constraints.

We estimated the parameters for each of the 7 transformed branches. Thus, N varied from time to time according to the number of original branches included in each group according to conversion table as follows: $N_1 = 4 \times 2 = 8$; $N_2 = 4 \times 2 = 8$; $N_3 = 10 \times 2 = 20$; $N_4 = 2 \times 2 = 4$; $N_5 = 3 \times 2 = 6$; $N_6 = 2 \times 2 = 4$; $N_7 = 5 \times 2 = 10$.

As far as the finite and discrete support for the matrix $\boldsymbol{\beta}' = [\gamma,\rho,\delta]' = \mathbf{Zp}$ and of the error term e is concerned, for both CES and Translog we have specified wide and symmetric supports as suggested by GME and GCE underlying theory in presence of scarce a priori information about the unknown parameters. Moreover, resort has been made to the information from SAM and to the economic theory, both used as a priori information (Ferrari and Manca, 2008).

3. Estimation results

The values of the parameters and of the elasticties of substitution estimated by using the GAMS software for both the first and the second nest of CES and Translog models are reported at Tables 1 and 2, and Tables 3 and 4 below respectively.

As the log-CES is non-linear in the parameters, we derived the asymptotic standard errors through a bootstrap procedure with which we were able to generate variances for all the parameters and corresponding pseudo t values. Instead, for Translog model, we derived the asymptotic standard error covariance matrix of $\hat{\beta}$ = $[\beta_0, \beta_1, \beta_{11}, \beta_{12}, \beta_{22}]$, $Var(\hat{\beta}) = \hat{\sigma}_L^2\ M(\hat{\beta})^{-1}(\mathbf{X'X})\ M(\hat{\beta})^{-1,}$ under an appropriate set of assumptions about the GCE solution to the general linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

Table 1 - GCE Estimates of the CES production function – VA - IC/L - K

	γ		ó		ρ	
Branch	**VA-IC**	**L-K**	**VA-IC**	**L-K**	**VA-IC**	**L-K**
	8.445 ***	1.223***	0.252	0.078	0.011	0.697***
Agriculture, hunting, forestry, etc.	(2.844)	(0.204)	(0.206)	(0.079)	(0.145)	(0.089)
	2.420 ***	0.701	0.167*	0.168	0.743***	-0.008
Industry	(0.444)	(0.435)	(0.087)	(0.195)	(0.012)	(0.144)
	6.396 ***	0.814***	0.494**	0.168***	1.024 ***	-0.011
Manufacture	(2.087)	(0.069)	(0.219)	(0.037)	(0.340)	(0.045)
	2.527 ***	0.881***	0.623 ***	0.524***	-0.107	-0.008
Energy and Construction	(0.305)	(0.329)	(0.138)	(0.117)	(0.072)	(0.023)
	2.250 ***	1.798**	0.814 ***	0.225***	0.080	0.175***
Services and Transport	(0.233)	(0.721)	(0.126)	(0.040)	(0.075)	(0.050)
	2.253 **	1.938***	0.704 ***	0.651	0.113***	-0.487***
Financial Intermediation, real estate, etc.	(0.952)	(0.000)	(0.123)	(0.568)	(0.018)	(0.159)
	8.592 ***	1.379***	0.849 ***	0.044	0.043	0.501***
Government, Education, Health, etc.	(0.713)	(0.000)	(0.074)	(0.084)	(0.307)	(0.126)

Notice: * significant at the 10% level. ** significant at the 5% level.*** significant at the 1% level.

Table 2 – HES/AES Elasticities of Substitution: σ= (1/1+ρ)

	Elasticity of substitution	
Branch	**VA-IC**	**K-L**
Agriculture, hunting, forestry and fishing - Mining	0.989	0.589
Industry	0.574	1.008
Manufacture	0.494	1.011
Energy and Construction	1.120	1.008
Services and Transport	0.926	0.851
Financial Intermediation, real estate and business activities	0.898	1.951
Government, Education, Health and other services	0.958	0.666

Table 3 - GCE Estimates of the Translog production function – VA - IC/L – K

	β_0		β_1		β_2		β_{11}		β_{12}		β_{22}	
Branch	**VA-IC**	**L-K**	**VA-IC**	**VA-IC**	**VA-IC**	**L-K**	**VA-IC**	**L-K**	**VA-IC**	**L-K**	**VA-IC**	**L-K**
A.	0.475	0.618**	0.825***	0.185	0.175	0.815***	0.017	-0.013	-0.008	0.002	-0.009	0.011
	(0.559)	(0.289)	(0.312)	(0.161)	(0.144)	(0.075)	(0.057)	(0.029)	(0.089)	(0.046)	(0.049)	(0.025)
I.	0.663**	0.534***	0.601***	0.503***	0.399**	0.497***	-0.103	-0.065	0.065	0.031	0.038	0.034
	(0.267)	(0.109)	(0.195)	(0.088)	(0.146)	(0.110)	(0.071)	(0.047)	(0.127)	(0.081)	(0.057)	(0.036)
M.	1.946***	0.597**	0.062	0.349**	0.938***	0.651***	0.002	-0.007	0.054	0.002	0.052	0.005
	(0.404)	(0.291)	(0.181)	(0.165)	(0.196)	(0.226)	(0.083)	(0.075)	(0.163)	(0.143)	(0.076)	(0.070)
E. & C.	1.189***	0.331***	0.845***	0.603***	0.155	0.497*	-0.120***	0.008***	0.067***	0.008	0.052	-0.016
	(0.086)	(0.034)	(0.260)	(0.104)	(0.272)	(0.272)	(0.003)	(0.001)	(0.005)	(0.005)	(0.270)	(0.108)
S. & T.	0.988***	0.601**	0.829	0.455	0.171	0.545	-0.010**	-0.045***	0.001	0.021***	0.009*	0.024
	(0.105)	(0.051)	(1.038)	(0.503)	(1.080)	(0.523)	(0.004)	(0.002)	(0.084)	(0.004)	(0.005)	(0.023)
F. I., etc.	0.090	0.140***	0.467*	0.518***	0.845	0.482	0.021***	-0.083***	0.001	0.048*	-0.022**	0.035***
	(0.062)	(0.035)	(0.262)	(0.150)	(0.920)	(0.528)	(0.004)	(0.002)	(0.031)	(0.028)	(0.010)	(0.006)
G., E., H., etc.	1.188	1.246*	0.902***	0.145*	0.098	0.855*	-0.029***	-0.078	0.003	0.014	0.025***	0.064
	(3.387)	(0.700)	(0.040)	(0.065)	(0.306)	(0.498)	(0.003)	(0.057)	(0.226)	(0.367)	(0.003)	(0.043)

Notice: * Significant at the 10% level. **Significant at the 5% level.*** Significant at the 1% level.

Table 4 – HES/AES Elasticities of Substitution: σ= (1/1+ρ)

	Elasticity of substitution	
Branch	**VA-IC**	**K-L**
Agriculture, hunting, forestry and fishing - Mining	1.388	1.082
Industry	2.272	1.474
Manufacture	1.564	1.060
Energy and Construction	2.340	1.132
Services and Transport	1.047	1.240
Financial Intermediation, real estate and business activities	1.024	1.767
Government, Education, Health and other services	1.105	0.775

4. Conclusion

GCE estimates of the parameters and elasticities of substitution for both CES and Translog production functions are satisfactorily significant and economically reliable on average. Non linearity and linearity of the functions seem not to affect the estimates and the results. The most relevant result of the research is that the estimation of the above parameters and elasticities of substitution is proved to be possible directly from SAM without introducing unrealistic hypotheses and methodological stretchings.

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