Hardy’s Condition in the Moment Problem for Probability Distributions

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In this paper we provide details about a not so well-known criterion for uniqueness of a probability distribution by its moments. The criterion is based on two old papers by G. H. Hardy (1917/1918), and to the best of our knowledge it is not explicitly described in the probabilistic literature. Hardy's condition has a simple form, often it is relatively easy to check and it allows us to derive new results or provide new proofs of known results.

In probabilistic terms we can describe Hardy's condition as follows: the square root of a nonnegative random variable has a moment generating function, which is weaker than Cramer's condition. The latter requires the existence of the moment generating function of the nonnegative random variable. We show that the constant (square root) in Hardy's condition is the best possible for the moment determinacy of the underlying distribution. Namely, it cannot be replaced by any smaller constant. Besides, two characterizations of Hardy's condition in terms of the moments of the underlying distribution are established. Finally, we apply Hardy’s criterion to study the uniqueness of a multivariate distribution by its multi-indexed moment sequence.

Key Words: Moments, Moment Problem, Hardy's Condition, Cramer's Condition, Carleman's Condition, Krein’s Condition