Theoretical Considerations for Multivariate Functional Data Analysis

Yoshiharu Sato
Hokkaido University, Sapporo, Japan. e-mail: yosiharu@ss.iij4u.or.jp

Abstract

Multivariate functional data is defined as an element of a direct sum of Hilbert spaces, $\boldsymbol{\mathcal{H}}^{(p)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_p$, where each $\mathcal{H}_k$ $(k = 1, 2, \ldots, p)$ is a real separable Hilbert space. In this paper, we consider a Gaussian measures on $\boldsymbol{\mathcal{H}}^{(p)}$ as its probability structure. That is, $\boldsymbol{\mathcal{H}}^{(p)}$-valued Gaussian random variable is defined for a measureable space.

Under the joint Gaussian probability measure, we will discuss the multivariate analysis just like the classical multivariate analysis. For this purpose, we shall extend the theory of finite dimensional multivariate normal distribution to $\boldsymbol{\mathcal{H}}^{(p)}$-valued random variables. Using the properties of $\boldsymbol{\mathcal{H}}^{(p)}$-valued Gaussian measure, it is shown that a concept of regression can be given by the conditional expectation and the concept of principal components is given by the use of eigen structure of covariance operator.

Keywords: conditional expectation, direct sum of Hilbert spaces, joint Gaussian measure, principal components.

1. Introduction

There have been many studies to date on "functional data analysis" commencing with the key book by Ramsay & Silverman ([8],2002). These functional data analysis have brought in many interested methods or procedures for the analysis of time series data and so on.

As a pioneering work for infinite dimensional multivariate analysis, Rao et al.([9],1963) have discussed discriminant analysis for two equivalent (not perpendicular) infinite dimensional Gaussian populations.

Main objective of multivariate analysis for the finite dimensional real random variables is an interpretation of the covariance structure through the several models. In traditional functional data analysis, the covariance between the pair of random variables is the same as the finite dimensional case. Then there are no essential differences in the covariance structures between the finite dimensional case and infinite case for finite samples.

On the other hand, Baker [2] has discussed the cross covariance operator, $\mathcal{R}_{ij}$, from $\mathcal{H}_j$ to $\mathcal{H}_i$. Using this concept, we can define the covariance operator of $\boldsymbol{\mathcal{H}}^{(p)}$-valued random variables and the joint Gaussian measure.

For multivariate functional data analysis, we discuss the properties of joint Gaussian measure and the concept of the regression is described by the conditional expectation in the same way of finite dimensional case.

Multivariate functional data has been discussed also by Berrendero et al. [3], in which they have discussed principal components. The framework of their discussion is required more general space than a Hilbert space in this paper. Here we discuss the model of principal components as an inner product in $\boldsymbol{\mathcal{H}}^{(p)}$.

2. Direct Sum Hilbert Space and Gaussian Measure

Let $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_p$ be the real separable Hilbert spaces. A direct sum of these Hilbert spaces is denoted by $\boldsymbol{\mathcal{H}}^{(p)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_p$. Here, we write the elements of $\boldsymbol{\mathcal{H}}^{(p)}$ by the column vector forms $\boldsymbol{u} = (u_1, u_2, \cdots, u_p)'$, $\boldsymbol{v} = (v_1, v_2, \cdots, v_p)'$, $u_a, v_a \in \mathcal{H}_a$, where the notation "′" denote the transpose of column vector.

The sum and the scalar multiplication for the elements of $\boldsymbol{\mathcal{H}}^{(p)}$ are defined by

$$\boldsymbol{u}' + \boldsymbol{v}' \equiv (u_1 + v_1, u_2 + v_2, \cdots, u_p + v_p), \quad \alpha\boldsymbol{u}' \equiv (\alpha u_1, \alpha u_2, \cdots, \alpha u_p),\ \alpha \in \mathbb{R}.$$

An inner product is defined for the pair of elements $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{\mathcal{H}}^{(p)}$ as follows:

$$[\boldsymbol{u}, \boldsymbol{v}]_{(p)} = \langle u_1, v_1 \rangle_1 + \langle v_2, v_2 \rangle_2 + \cdots + \langle u_p, v_p \rangle_p,$$

where $\langle \cdot, \cdot \rangle_a$ is denoted the inner product on $\mathcal{H}_a$. Under these definition, $\boldsymbol{\mathcal{H}}^{(p)}$ becomes Hilbert space. For the element $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}^{(p)}$ a norm is introduced by $\|\boldsymbol{u}\|_{(p)} = \sqrt{[\boldsymbol{u}, \boldsymbol{u}]_{(p)}}$. If the each Hilbert space $\mathcal{H}_a$ has completeness, it will be shown that the Hilbert space $\boldsymbol{\mathcal{H}}^{(p)}$ has also completeness.

First we shall consider a real separable Hilbert space $\mathcal{H}$.

Definition 2.1. (Random variable) An $\mathcal{H}$-valued random variable is an $\mathcal{H}$-valued strongly $\mathbb{P}$-measureable ([6]) function X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

All integral of $\mathcal{H}$-valued random variable will be Bochner-integrals. For integrable random variable X, mean value is defined by $\mathbb{E}X \equiv \int_\Omega X d\mathbb{P}$.

Definition 2.2. (Borel probability measure) The distribution of an $\mathcal{H}$-valued random variable X is the Borel probability measure μ_X on $\mathcal{H}$ defined by $\mu_X(B) \equiv \mathbb{P}\{X \in B\}$, $B \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is Borel σ-algebra of $\mathcal{H}$, which is denoted by $\Gamma \equiv \mathcal{B}(\mathcal{H})$.

Definition 2.3. (Fourier transform) The Fourier transform of random variable $X : \Omega \to \mathcal{H}$ is denoted by Fourier transform of Borel probability measure μ_X defined by

$$\hat{\mu}_X(u) \equiv \mathbb{E}\exp(-i\langle X, u \rangle) = \int_{\mathcal{H}} \exp(-i\langle x, u \rangle) d\mu_X(x)$$

Theorem 2.1. (Parthasarathy, 1967[7]) Let X_1 and X_2 be $\mathcal{H}$-valued random variables whose Fourier transforms are equal, i.e. $\hat{\mu}_{X_1}(u) = \hat{\mu}_{X_2}(u)$ for all $u \in \mathcal{H}$. Then X_1 and X_2 are identical distributed.

Definition 2.4. (Gaussian random variable) An $\mathcal{H}$-valued random variable X is Gaussian if the real random variable $\langle X, u \rangle$ is Gaussian for all $u \in \mathcal{H}$. The Borel probability measure μ_X is called Gaussian measure on $(\mathcal{H}, \Gamma)$.

Theorem 2.2 (Fernique,1970[5]) Let X be an $\mathcal{H}$-valued Gaussian random variable, there exist a constant $\beta > 0$ such that $\mathbb{E}\exp(\beta\|X\|^2) < \infty$.

By the Fernique theorem, it will suffice to show $\mathbb{E}\|X\|^p < \infty$ for all $1 \leq p < \infty$. Also X is considered to be a μ_X-Bochner integrable. Then for all $u \in \mathcal{H}$ we can denote $\langle \mathbb{E}X, u \rangle = \mathbb{E}\langle X, u \rangle$. By the Riesz representation theorem, there exist an element such that $\mathbb{E}X = m \in \mathcal{H}$. We call m the mean element of X with respect to μ_X.

Definition 2.5. A bonded operator $\mathcal{R} : \mathcal{H} \to \mathcal{H}$ is called (i) positive, if $\langle \mathcal{R}u, u \rangle \geq 0$ for all $u \in \mathcal{H}$; and (ii) symmetric, if $\langle \mathcal{R}u, v \rangle = \langle \mathcal{R}v, u \rangle$ for all $u, v \in \mathcal{H}$.

Theorem 2.3 (Parthasarathy,1967[7])For $\mathcal{H}$-valued random variable X, the Borel measure μ_X is Gaussian if and only if there exist a positive symmetric operator $\mathcal{R}$ on $\mathcal{H}$ uniquely and Fourier transform of μ_X is given by

$$\hat{\mu}_X = \mathbb{E}\exp(-i\langle X, u\rangle) = \exp\left(i\langle m, u\rangle - \frac{1}{2}\langle \mathcal{R}u, u\rangle\right), \quad u \in \mathcal{H} \tag{1}$$

The operator $\mathcal{R}$ is called covariance operator of $\mathcal{H}$-valued random variable X.

On the other hand, the Neumann-Schatten product is defined by $(f \otimes g)h \equiv \langle h, g\rangle f$ for all $h \in \mathcal{H}$. Using this operator, the covariance operator $\mathcal{R}$ is described as follows;

$$\langle \mathcal{R}u, v\rangle = \int_{\mathcal{H}} \langle \{(x-m) \otimes (x-m)\}u, v\rangle \, d\mu_X(x)$$

Then we may describe that the covariance operator is given by the expectation of the Neumann-Schatten product, i.e. $\mathbb{E}\{(X-m)\otimes(X-m)\} = \mathcal{R}$.

Next we shall discuss a joint Gaussian measure on the direct sum Hilbert space $\boldsymbol{\mathcal{H}}^{(p)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_p$. Let $\Gamma_a = \mathcal{B}(\mathcal{H}_a)$ be a Borel σ-field on $\mathcal{H}_a$ derived from the norm on $\mathcal{H}_a$, and $\boldsymbol{\Gamma}_{(p)} \equiv \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_p$ denote the Borel σ-field generated by the measurable rectangles $A_1 \times A_2 \times \cdots \times A_p$, $A_a \in \Gamma_a (a = 1, 2, \ldots, p)$. Let $\boldsymbol{X}_{(p)} \equiv (X_1, X_2, \cdots, X_p)'$ be a $\boldsymbol{\mathcal{H}}^{(p)}$-valued random variable and $\boldsymbol{\mu}_{(p)}$ be a Borel probability measure on $(\boldsymbol{\mathcal{H}}^{(p)}, \boldsymbol{\Gamma}_{(p)})$. $\boldsymbol{\mu}_{(p)}$ will be called a joint measure of $(X_1, X_2, \cdots, X_p)$.

Since $(\boldsymbol{\mathcal{H}}^{(p)}, [\cdot, \cdot]_{(p)})$ is a real separable Hilbert space, we can define a joint Gaussian measure from the Definition 2.4.

Definition 2.6. (Joint Gaussian random variables) An $\boldsymbol{\mathcal{H}}^{(p)}$-valued random variable $\boldsymbol{X}_{(p)} = (X_1, X_2, \cdots, X_p)'$ is a joint Gaussian if the real random variable $[\boldsymbol{X}_{(p)}, \boldsymbol{u}]_{(p)}$ is Gaussian for all $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}^{(p)}$. The Borel probability measure $\boldsymbol{\mu}_{(p)}$ is called a joint Gaussian measure on $(\boldsymbol{\mathcal{H}}^{(p)}, \boldsymbol{\Gamma}_{(p)})$.

The mean elements of $\boldsymbol{X}_{(p)} = (X_1, X_2, \cdots, X_p)'$ is given by $\boldsymbol{m}_{(p)} = (m_1, m_2, \cdots, m_p)'$, where each m_a is the mean element of X_a with respect to μ_a on $(\mathcal{H}_a, \boldsymbol{\Gamma}_{(p)}/\Gamma_a)$.

The covariance operator of X_a with respect to μ_a on $(\mathcal{H}_a, \boldsymbol{\Gamma}_{(p)}/\Gamma_a)$ is given by

$$\langle \mathcal{R}_a u_a, v_a\rangle_a = \int_{\mathcal{H}_a} \langle (x_a - m_a), u_a\rangle_a \langle (x_a - m_a), v_a\rangle_a d\mu_a(x_a)$$

And Baker[2] has discussed the cross-covariance operator of Gaussian measure μ_{ab} on $(\mathcal{H}_a \oplus \mathcal{H}_b, \boldsymbol{\Gamma}_{(p)}/\Gamma_a \times \Gamma_b)$ which is defined by the operator from $\mathcal{H}_b$ to $\mathcal{H}_a$, $\mathcal{R}_{ab} : \mathcal{H}_b \to \mathcal{H}_a$, where

$$\langle \mathcal{R}_{ab} v_b, u_a\rangle_a = \int_{\mathcal{H}_a \oplus \mathcal{H}_b} \langle (x_a - m_a), u_a\rangle_a \langle (x_b - m_b), v_b\rangle_b d\mu_{ab}(x_a, x_b).$$

Then we assume an operator matrix

$$\boldsymbol{\mathcal{R}}_{(p)} \equiv \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_{12} & \cdots & \mathcal{R}_{1p} \\ \mathcal{R}_{21} & \mathcal{R}_2 & \cdots & \mathcal{R}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_{p1} & \mathcal{R}_{p2} & \cdots & \mathcal{R}_p \end{bmatrix}.$$

The operator $\boldsymbol{\mathcal{R}}_{(p)}$ works just like the matrix operation on the element $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}^{(p)}$ as follows:

$$\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{u} = \begin{bmatrix} \mathcal{R}_1 u_1 + \mathcal{R}_{12} u_2 + \cdots + \mathcal{R}_{1p} u_p \\ \mathcal{R}_{21} u_1 + \mathcal{R}_2 u_2 + \cdots + \mathcal{R}_{2p} u_p \\ \vdots \\ \mathcal{R}_{p1} u_1 + \mathcal{R}_{p2} u_2 + \cdots + \mathcal{R}_p u_p \end{bmatrix} \in \boldsymbol{\mathcal{H}}^{(p)}$$

Using these notations the covariance operator of $\boldsymbol{X}_{(p)}$ is given by

$$[\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{u}, \boldsymbol{v}]_{(p)} = \int_{\boldsymbol{\mathcal{H}}^{(p)}} [(\boldsymbol{x}_{(p)} - \boldsymbol{m}_{(p)}), \boldsymbol{u}]_{(p)} [(\boldsymbol{x}_{(p)} - \boldsymbol{m}_{(p)}), \boldsymbol{v}]_{(p)} d\boldsymbol{\mu}_{(p)}(\boldsymbol{x}_{(p)}), \ \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{\mathcal{H}}^{(p)} \qquad (2)$$

For the pair of elements $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{\mathcal{H}}^{(p)}$, a Neumann-Schatten product $(\boldsymbol{u} \otimes \boldsymbol{v}) : \boldsymbol{\mathcal{H}}^{(p)} \to \boldsymbol{\mathcal{H}}^{(p)}$ is defined by $(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} \equiv [\boldsymbol{w}, \boldsymbol{v}]_{(p)}\boldsymbol{u}$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{H}}$. Then the covariance operator $\boldsymbol{\mathcal{R}}_{(p)}$ is denoted by the expectation of the following Neumann-Schatten product

$$\mathbb{E}\{(\boldsymbol{X}_{(p)} - \boldsymbol{m}_{(p)}) \otimes (\boldsymbol{X}_{(p)} - \boldsymbol{m}_{(p)})\} = \boldsymbol{\mathcal{R}}_{(p)}.$$

Since $(\boldsymbol{\mathcal{H}}^{(p)}, [\cdot, \cdot]_{(p)})$ is a real separable Hilbert space, we get the following

Theorem 2.4. For $\boldsymbol{\mathcal{H}}^{(p)}$-valued random variable $\boldsymbol{X}_{(p)}$, the Borel measure $\boldsymbol{\mu}_{(p)}$ is Gaussian if and only if there exists a positive symmetric operator $\boldsymbol{\mathcal{R}}_{(p)}$ on $\boldsymbol{\mathcal{H}}^{(p)}$ and mean element $\boldsymbol{m}$ uniquely and Fourier transform of $\boldsymbol{\mu}_{(p)}$ is given by

$$\hat{\boldsymbol{\mu}}_{(p)} = \mathbb{E}\exp(-i[\boldsymbol{X}_{(p)}, \boldsymbol{u}]_{(p)}) = \exp\left(i[\boldsymbol{m}_{(p)}, \boldsymbol{u}]_{(p)} - \frac{1}{2}[\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{u}, \boldsymbol{u}]_{(p)}\right), \ \boldsymbol{u} \in \boldsymbol{\mathcal{H}}^{(p)} \qquad (3)$$

Theorem 2.5. Let $\boldsymbol{X}_{(p)}$be a $\boldsymbol{\mathcal{H}}^{(p)}$-valued Gaussian random variable with a mean element $\boldsymbol{m}_{(p)}$ and a covariance operator $\boldsymbol{\mathcal{R}}_{(p)}$. If $\boldsymbol{\mathcal{T}}$ is a bounded operator on $\boldsymbol{\mathcal{H}}^{(p)}$, then $\boldsymbol{\mathcal{T}}\boldsymbol{X}_{(p)}$ is also $\boldsymbol{\mathcal{H}}^{(p)}$-valued Gaussian random variable with the mean element $\boldsymbol{\mathcal{T}}\boldsymbol{m}_{(p)}$ and the covariance operator $\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{\mathcal{T}}^*$, where $\boldsymbol{\mathcal{T}}^*$is an adjoint operator on $\boldsymbol{\mathcal{H}}^{(p)}$.

Definition 2.7. Let X_a $(a = 1, 2, \ldots, p)$ be a $\mathcal{H}_a$-valued random variable and $\mu_a(w_a)$ be a its probability measure. If $\boldsymbol{X}_{(p)} = (X_1, X_2, \cdots, X_p)'$ is $\boldsymbol{\mathcal{H}}^{(p)}$-valued random variable with joint probability Borel measure $\boldsymbol{\mu}_{(p)}$, then the set of random variables are said to be mutually independent if

$$\hat{\boldsymbol{\mu}}_{(p)}(\boldsymbol{w}) = \hat{\mu}_1(w_1)\hat{\mu}_2(w_2)\cdots\hat{\mu}_p(w_p),$$

for any $\boldsymbol{w} = (w_1, w_2, \cdots, w_p)' \in \boldsymbol{\mathcal{H}}^{(p)}$, where $\hat{\mu}_a(w_a)$ is Fourier transform of $\mu_a(w_a)$ and $\hat{\boldsymbol{\mu}}_{(p)}(\boldsymbol{w})$ is also Fourier transform of $\boldsymbol{\mu}_{(p)}$.
We suppose that $\boldsymbol{\mathcal{H}}^{(p)}$-valued random variable $\boldsymbol{X}_{(p)}$ is divided into two groups $\boldsymbol{X}_{(1)} \equiv (X_1, \cdots, X_r)'$ and $\boldsymbol{X}_{(2)} \equiv (X_{r+1}, \cdots, X_p)'$ and $\boldsymbol{X}_{(1)}$ and $\boldsymbol{X}_{(2)}$ are also joint probability Borel measure $\boldsymbol{\mu}_{(1)}$ and $\boldsymbol{\mu}_{(2)}$, respectively. The set $\boldsymbol{X}_{(1)}$ is said to be independent of the set $\boldsymbol{X}_{(2)}$ if

$$\hat{\boldsymbol{\mu}}_{(p)}(\boldsymbol{w}) = \hat{\boldsymbol{\mu}}_{(1)}(\boldsymbol{w}_{(1)})\hat{\boldsymbol{\mu}}_{(2)}(\boldsymbol{w}_{(2)}),$$

where $\boldsymbol{w}_{(1)} = (w_1, \cdots, w_r)', \boldsymbol{w}_{(2)} = (w_{r+1}, \cdots, w_p)'$, $\boldsymbol{w} = (\boldsymbol{w}_{(1)}, \boldsymbol{w}_{(2)})' \in \boldsymbol{\mathcal{H}}^{(p)}$, and $\hat{\boldsymbol{\mu}}_{(1)}$, $\hat{\boldsymbol{\mu}}_{(2)}$ are Fourier transform of $\boldsymbol{\mu}_{(1)}$ and $\boldsymbol{\mu}_{(2)}$, respectively.

Based on this definition, we get the following

Theorem 2.6. Let $\boldsymbol{X}_{(p)} = (X_1, X_2, \cdots, X_p)'$ be a $\boldsymbol{\mathcal{H}}^{(p)}$-valued joint Gaussian random variables with mean elements $\boldsymbol{m}_{(p)} = (m_1, m_2, \cdots, m_p)'$ and covariance operator $\boldsymbol{\mathcal{R}}_{(p)}$. A necessary and sufficient condition that one subset of random variables and the subset consisting of the remaining variables be independent is that each cross-covariance operator of a variable from one set and a variable from the other set be null operator.

3. Conditional Expectation and Regression

Hereafter we assume that the covariance operators of any Gaussian random variable and joint Gaussian random variables are invertible, that is, the kernel of covariance operator is always constructed of a zero element in corresponding Hilbert space.

For the partition of the Gaussian random variable $\boldsymbol{X}_{(p)}$ just like the definition 2.7, we assume that the mean elements and covariance operator are divided into as follows:

$$\boldsymbol{X}_{(p)} = \begin{bmatrix} \boldsymbol{X}_{(1)} \\ \boldsymbol{X}_{(2)} \end{bmatrix}, \quad \boldsymbol{m}_{(p)} = \begin{bmatrix} \boldsymbol{m}_{(1)} \\ \boldsymbol{m}_{(2)} \end{bmatrix}, \quad \boldsymbol{\mathcal{R}}_{(p)} = \begin{bmatrix} \boldsymbol{\mathcal{R}}_{(1)(1)} & \boldsymbol{\mathcal{R}}_{(1)(2)} \\ \boldsymbol{\mathcal{R}}_{(2)(1)} & \boldsymbol{\mathcal{R}}_{(2)(2)} \end{bmatrix},$$

where $\boldsymbol{\mathcal{R}}_{(a)(b)} \equiv \mathbb{E}\{(\boldsymbol{X}_{(a)} - \boldsymbol{m}_{(a)}) \otimes (\boldsymbol{X}_{(b)} - \boldsymbol{m}_{(b)})\}$, $a, b = 1, 2$. If $\boldsymbol{X}_{(1)}$ and $\boldsymbol{X}_{(2)}$ are not independent each other, we shall consider a linear transformation in $\boldsymbol{\mathcal{H}}^{(p)}$ as follows by the use of linear bounded operator $\boldsymbol{\mathcal{M}}$ from $\boldsymbol{\mathcal{H}}^{(p-r)}$ to $\boldsymbol{\mathcal{H}}^{(r)}$,

$$\boldsymbol{Y}_{(1)} = \boldsymbol{X}_{(1)} + \boldsymbol{\mathcal{M}}\boldsymbol{X}_{(2)}, \qquad \boldsymbol{Y}_{(2)} = \boldsymbol{X}_{(2)} \tag{4}$$

choosing $\boldsymbol{\mathcal{M}}$ so that the random variables $\boldsymbol{Y}_{(1)}$ are independent of $\boldsymbol{Y}_{(2)}$. By the same way as the classical multivariate analysis [1], we get $\boldsymbol{\mathcal{M}} = -\boldsymbol{\mathcal{R}}_{(1)(2)}\boldsymbol{\mathcal{R}}_{(2)(2)}^{-1}$. Using this result, the expression (4) is given by

$$\boldsymbol{Y}_{(p)} = \begin{bmatrix} \boldsymbol{Y}_{(1)} \\ \boldsymbol{Y}_{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{I}}_{(r)} & -\boldsymbol{\mathcal{R}}_{(1)(2)}\boldsymbol{\mathcal{R}}_{(2)(2)}^{-1} \\ \boldsymbol{\mathcal{O}} & \boldsymbol{\mathcal{I}}_{(p-r)} \end{bmatrix} \boldsymbol{X}_{(p)} \equiv \boldsymbol{\mathcal{T}}\boldsymbol{X}_{(p)}.$$

where $\boldsymbol{\mathcal{I}}_{(r)}$ and $\boldsymbol{\mathcal{I}}_{(p-r)}$ are the identity operators. Then the mean elements of $\boldsymbol{Y}_{(p)}$ is given by $\boldsymbol{\mathcal{T}}\boldsymbol{m}_{(p)}$ and the covariance operator of $\boldsymbol{Y}_{(p)}$ is given by $\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{\mathcal{T}}^*$. Since $\boldsymbol{Y}_{(1)}$ and $\boldsymbol{Y}_{(2)}$ are independent, we get

$$\mathbb{E}(\boldsymbol{X}_{(1)}|\boldsymbol{x}_2) = \boldsymbol{m}_{(1)} + \boldsymbol{\mathcal{R}}_{(1)(2)}\boldsymbol{\mathcal{R}}_{(2)(2)}^{-1}(\boldsymbol{x}_{(2)} - \boldsymbol{m}_{(2)}) \equiv \boldsymbol{\eta}(\boldsymbol{x}_{(2)})$$

If we denote the covariance operator of $(\boldsymbol{X}_{(1)}|\boldsymbol{x}_2)$ by $\boldsymbol{\mathcal{R}}_{(1)(1)\cdot(2)}$, then it should be given by $\boldsymbol{\mathcal{R}}_{(1)(1)\cdot(2)} = \boldsymbol{\mathcal{R}}_{(1)(1)} - \boldsymbol{\mathcal{R}}_{(1)(2)}\boldsymbol{\mathcal{R}}_{(2)(2)}^{-1}\boldsymbol{\mathcal{R}}_{(2)(1)}$. Along with usual finite dimensional multivariate analysis ([1]), $\boldsymbol{\eta}(\boldsymbol{x}_{(2)})$ is called regression of $\boldsymbol{X}_{(1)}$ on $\boldsymbol{x}_{(2)}$. and $\boldsymbol{\mathcal{R}}_{(1)(2)}\boldsymbol{\mathcal{R}}_{(2)(2)}^{-1}$ is defined as regression coefficient operator.

4. Principal Components

Let $\boldsymbol{X}_{(p)} = (X_1, X_2, \cdots, X_p)'$ be a $\boldsymbol{\mathcal{H}}^{(p)}$-valued Gaussian random variables and $\boldsymbol{\mu}_{(p)}$ be a Borel joint Gaussian probability measure. Now we shall consider an inner product

$$Z = [\boldsymbol{X}_{(p)}, \boldsymbol{\alpha}]_{(p)}, \quad \|\boldsymbol{\alpha}\| = 1, \ \boldsymbol{\alpha} \in \boldsymbol{\mathcal{H}}^{(p)}$$

Since $\boldsymbol{X}_{(p)}$ is joint Gaussian, Z is a real Gaussian random variable. Hereafter we assume that $\mathbb{E}(\boldsymbol{X}_{(p)}) = \boldsymbol{0}$, without loss of generality. By this assumption, $\mathbb{E}(Z) = [\mathbb{E}(\boldsymbol{X}_{(p)}), \boldsymbol{\alpha}]_{(p)} = 0$, then the variance of Z is given by $\mathbb{V}(Z) = \mathbb{E}\{[\boldsymbol{X}_{(p)}, \boldsymbol{\alpha}]_{(p)}[\boldsymbol{X}_{(p)}, \boldsymbol{\alpha}]_{(p)}\} = \left[\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{\alpha}, \boldsymbol{\alpha}\right]_{(p)}$. On the other hand, let $\lambda \in \mathbb{R}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{H}}^{(p)}$ are the eigenvalue and eigenvector of $\boldsymbol{\mathcal{R}}_{(p)}$, respectively, we get the following relation $\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{u} = \lambda\boldsymbol{u}, \quad \|\boldsymbol{u}\| = 1$. Then if we put $\boldsymbol{\alpha} = \boldsymbol{u}$, it follows $\mathbb{V}(Z) = [\boldsymbol{\mathcal{R}}_{(p)}\boldsymbol{u}, \boldsymbol{u}]_{(p)} = \lambda[\boldsymbol{u}, \boldsymbol{u}]_{(p)} = \lambda$, that is, the variance of Z is given by the eigenvalue λ of $\boldsymbol{\mathcal{R}}_{(p)}$. In order to maximize the variance of Z, we may take the coefficient element $\boldsymbol{\alpha}$ as an eigenvector $\boldsymbol{u}_1$ corresponding the largest eigenvalue λ_1, i.e. $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Then we put $Z_1 = [\boldsymbol{X}_{(p)}, \boldsymbol{u}_1]_{(p)}$ and Z_1 is called the first principal component. For each λ_k $(k = 2, 3, \ldots)$ we put again $Z_2 = [\boldsymbol{X}_{(p)}, \boldsymbol{u}_2]_{(p)} \quad \cdots \quad Z_k = [\boldsymbol{X}_{(p)}, \boldsymbol{u}_k]_{(p)}$ by the use of corresponding eigenvector $\boldsymbol{u}_k$ $(k = 2, 3, \ldots)$. Z_k is called k-th principal component. Since covariance operator $\boldsymbol{\mathcal{R}}_{(p)}$ is a symmetric positive operator of trace class, we get the following relation

$$\mathbb{E}(Z_i Z_j) = \lambda_i \delta_{ij}.$$

Since Z_k takes the real values, the most interesting point of these principal components will be describe the multivariate functional data by the finite dimensional real space if we take finite number of principal components.

5. Concluding Remarks

A real separable Hilbert space will be considered an infinite dimensional Euclidean space in some aspects. Then the statistical data analysis under the framework of Gaussian measure on a Hilbert space seems to gain no revelation compaired with classical multivariate analysis although the dimension of the data could be extended.

In this paper, we will investigate the possibility and the theoretical background for multivariate functional data analysis using joint Gaussian measure on direct sum of finite number of Hilbert spaces.

The author is also interested in the generalization of the concept of discriminant analysis by Rao et al. [9] to multivariate functional data.

References

[1] Anderson, T.W.(1984) *An Introduction to Multivariate Statistical Analysis*, Second Edition, John Wiley & Sons, New York.

[2] Baker, C.R.(1973)" Joint Measure and Cross-Covariance Operators ", *Transactions of the American Mathematical Society*, **186**, 273–289.

[3] Berrendero, J.R., Justel, A. & Svarc, M.(2011) "Principal components for Multivariate Functional Data", *Computational Statistical and Data Analysis*, **55**, 2619–2634.

[4] Bogachev, V.I.(1998) *Gaussian Measures*, Mathematical Surveys and Monographs, **62**, American Mathematical Society. (Translated from the original Russian manuscript by Vladimir I. Bogachev)

[5] Fernique, X.(1970) "Intégrabilité des vecteurs gaussiens", C.R.Acad Sci. Paris Sé. A-B **270** A1698–A1699.

[6] van Neerven, J.M.A.M.(2007) "Stochastic evolution equations", *ISEM Lecture Notes* 2007/08. fa.its.tudelft.nl/neerven/publications/papers/ISEM.pdf.

[7] Parthasarathy, K.R.(1967) *Probability measures on metric spaces*, Probability and mathematical statistics, a series of monographs and textbooks, Academic Press, New York.

[8] Ramsay, J.O. & and Silverman, B.W.(2002) *Applied Functional Data Analysis : Methods and Case Studies*, Springer, New York.

[9] Rao, C.R. & Varadarajan, V.S.(1963) "Discrimination of Gaussian Processes", *Sankhya*, Series A, **25**, 303–330.