Realized skewness at high frequency and link to conditional market premium

Zhi Liu[b,c], Kent Wang[a], Junwei Liu[a]

[a] *The Wangyanan Institute for Studies in Economics, Xiamen University, China*
[b] *Department of Mathematics, University of Macau, Macau, China*
[c] *Corresponding author: Zhi LIU, Email: liuzhi@umac.mo*

Abstract

We propose a reliable new estimator for realized skewness which is robust to microstructure noise at ultra-high frequency level. Asymptotic theory for the new estimator has been de- rived. Simulation example verifies its superior performance. We apply the new estimator to tick data of the S&P 500 index for forecasting equity premium in the U.S. market from 1990-2011 and find that it has significant forecast-ability both in-sample and out-of-sample. We also show that the new skewness measure plus the variance risk premium provides right decomposition for the skewness risk. We thus provide evidence that realized skewness links to conditional market premium.

Keywords: , High-frequency, Jump, Noise, Skewness, Stock return prediction

1. Introduction

Traditional approaches of asset pricing concentrate mainly on the first and second moment of returns. But there is considerable evidence that the third moment may also be important. Barro (2009) argues that the inclusion of the possibility of rare disasters (extreme negative skewness) in an otherwise reasonably standard representative-agent economy, generates equity premia and risk free rates that accord with observation. The empirical literature for skewness going back to Kraus and Litzenberger (1976), and including more recently Das and Sundaram (1997), Harvey and Siddique (2000), Ang et al. (2006), Chen et al. (2006) and Xing et al. (2010), suggests that the asymmetry of the returns distribution both for individual stocks and for the equity market as a whole is important for asset pricing and investment management.

Different skewness measures have been proposed in the literature but it is generally admitted that skewness is hard to measure precisely. Major existing skewness proxies with theoretical foundations include risk neutral skewness measure advocated by Bakshi et al., realized skewness measures used in such as Amaya et al. (2011)(ACJV (2011) hereafter)[1], Neuberger (2012) and skewness premium by combining implied and realized skewness. The difficulty for measuring skewness precisely

[1] Note that the skewness measure in ACJV (2011) is standard skewness measure according to skewness definition but uses high frequency data. In the current study, ACJV (2011) is one example we use among many who apply such measure in empirical works.

lies in the fact that people know little about the determinants or sources of skewness and the understanding of the issue comes from limited empirical evidence which are mainly based on hypothesis or even speculations[2].

The recent advent of readily-available high-frequency data, however, has spurred a renewed interest into alternative ways for more accurately estimating skewness. It is generally found that measures based on high frequency data can improve the estimation and result in far better empirical performances (see, e.g. ACJV 2011). It is believed that one needs to be able to exploit high frequency data to improve the estimation of skewness if one seeks to understand the dynamics of skewness of period returns Neuberger (2012). However, the realized skewness measures proposed based on high frequency data are mostly just proxies with limited validity due to lack of theoretical foundation and empirical considerations. For example, the realized skewness measure appeared in ACJV (2011), which is in terms of the ratio of the third cumulant and the 1.5th power of the second cumulant, has not considered any of the common factors in the high frequency estimation such as microstructure noise and jumps etc. They neither provide any theoretical proof for the unbiasedness and consistency of the proposed empirical estimator.Neuberger (2012) provides a very interesting ex-post estimator for skewness and we believe it is by far the most theoretically justified measure for realized skewness. But Neuberger (2012) only provides unbiasedness proof for the estimator without the consistency validation. In addition, due to model setting, his measure does not accommodate important factors such as noise in the high frequency data and hence can not be applied to the tick data if needed. This leads us to believe that most of the current empirical measures for ex-post skewness based on high frequency data are in fact rough indicators for the "true" skewness in high frequency. There is great need to improve those estimators based on theoretical and empirical considerations, which is what we propose to do in the current study.

Based on the existing realized skewness measure, we derive the asymptotic properties for it and then develop our own measure of realized skewness accounting for microstructure noise effect in the data. Simulation study is also performed to investigate finite sample properties of the new measure and it is found that the proposed new estimator can produce far more accurate estimation for the realized third moment that is quite close to the ideal situation. We then apply the new method to the tick data of the S&P 500 index from 1990-2011 in forecasting near future excess equity market returns. We find that the new estimator for realized skewness based on tick data significantly forecasts one-month-ahead excess equity market returns and such result is obtained controlling for commonly used stock return predictors in the literature including the powerful variance risk premium (VRP). The forecast ability of our skewness measure remains significant at 5% in the out-of-sample prediction and shows considerable economic importance. This sheds new light on the recognized puzzle that skewness measures only have cross-sectional pricing effects while little time series performance has been documented before. The result thus suggests that realized skewness is an effective predictor for stock market returns and

[2]However, it is not too difficult to understand at intuitive level that asset return skewness is caused by the jumps in the returns along with the leverage effect. Such point of view can be found from works such as Das and Sundaram (1997) and Neuberger (2012).

controlling for noise can realistically reveal such effectiveness. In a further attempt to explore the sources of documented predictability of the realized skewness in the current study, we find the new realized skewness measure subsumes to a certain degree the market momentum effect in the short run. The finding that both our new realized skewness measure and the variance risk premium are significant in predicting aggregate stock market premium indicates that the new measure provides an independent channel for stock return prediction and we supply the literature with a clean decomposition of the skewness risk into jump and leverage effect components. We therefore shed new light on both the time series pricing performance of the skewness and the hypothesis regarding the sources of the skewness risk.

We organize the rest of the paper as following. Section 2 provides theoretical discussions for deriving asymptotic properties of the existing high frequency based estimator for skewness and kurtosis as in ACJV (2011) as well as theories for developing our new skewness measure that accounts for microstructure noise; Section 3 provides discussions for the simulation studies; Section 4 presents empirical illustrations for applying the new skewness measures in forecasting excess S&P 500 index returns; Section 5 provides further discussion for the results obtained and Section 6 concludes. We postpone all the necessary theoretical proof and justification for our methodology to the Appendix.

2. Methodology

2.1. The Model and Estimators

We define two processes $\{X_t,\ t \geq 0\}$ and $\{Y_t,\ t \geq 0\}$, both are adapted to the filtered probability space (Ω, F, F_t, P) as follows:

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s,\ Y_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{s \leq t} \Delta_s Y, \tag{1}$$

where $\{b_t, 0 \leq t \leq T\}$ is adapted locally bounded processes, $\{\sigma_t, 0 \leq t \leq T\}$ is an deterministic càdlàg volatility process. $\Delta_s Y = Y_s - Y_{s-}$ is the jump of Y at time s, if there is no jumps, it is simple zero. We assume that the jump of Y arrives via a finite activity jump process, which includes commonly used compound Poisson process as a special case. We observe the processes at some discrete time points t_i, $0 = t_0 \leq t_i \leq t_n = T$, that is, we have the increments as $\Delta Z_i = Z_{t_{i+1}} - Z_{t_i}$ for either $Z = X$ or $Z = Y$. Hence, in the following context, we use Z referring to either as X or Y, namely, when $Z = X$, it has continuous path, $Z = Y$ means there is jumps in the path.

2.2. Microstructure-noise free estimator

In this section, we investigate the properties of the estimator proposed by ACJV (2011) under both X and Y. We assume that the log price follows the process Z, and then the increments correspond to the log returns for each firm. ACJV (2011) proposed the realized daily skewness and realized kurtosis as the following:

$$RDSkew = \frac{\sqrt{n}\sum_{i=1}^{n}(\Delta Z_i)^3}{RDVar_T^{3/2}}, \quad RDKurt = \frac{n\sum_{i=1}^{n}(\Delta Z_i)^4}{RDVar_T^2}, \tag{2}$$

where $RDVar_t$ defines well-known realized volatility $RDVar_T = \sum_{i=1}^{n}(\Delta Z_i)^2$. We now study the asymptotic properties of the estimators.

THEOREM 1. *Z defines a Itô process, and $\Delta Z_i = Z_{t_{i+1}} - Z_{t_i}$. Then*

1. *When $Z = X$, we have*

$$\sum_{i=1}^{n}(\Delta Z_i)^2 \to_P \int_0^T \sigma_t^2 dt, n\sum_{i=1}^{n}(\Delta Z_i)^3 \to_D N, n\sum_{i=1}^{n}(\Delta Z_i)^4 \to_P 3\int_0^T \sigma_t^4 dt. \quad (3)$$

Where, N is a random variable with normal distribution, and $\to_D$ defines the convergence in distribution and $\to_P$ is the convergence in probability.

2. *When $Z = Y$, we have*

$$\sum_{i=1}^{n}(\Delta Z_i)^2 \to_P \int_0^T \sigma_t^2 dt + \sum_{t\le T}(\Delta_t Y)^2, \quad (4)$$

$$\sqrt{n}\sum_{i=1}^{n}(\Delta Z_i)^3 \to_P \infty, \ n\sum_{i=1}^{n}(\Delta Z_i)^4 \to_P \infty. \quad (5)$$

REMARK 2. From the above results, we have that asymptotically, both of $RDSkew$ and $RDKurt$ are not well-defined in the jump-diffusion model (tend to infinity asymptotically). In the diffusion model, the $RDSkew$ is not reasonable since $RDSkew \to_P 0$. Considering of jump is accepted both theoretically and empirically, instead of pursuing more sophistical estimator for the diffusion model, we aim to propose some reasonable estimators for jump-diffusion model.

To deal with this problem, we propose to use the following new estimators for skewness and kurtosis:

$$J - RDSkew := \frac{\sum_{i=1}^{n}(\Delta Z_i)^3}{RDVar_t^{3/2}}, \quad J - RDKurt := \frac{\sum_{i=1}^{n}(\Delta Z_i)^4}{RDVar_t^2}. \quad (6)$$

Here we use J to emphasize the “jumps”. The following theorem specifically describes the asymptotic performance of the estimators proposed above.

THEOREM 3. *$Z = Y$ is defined as in (1), we have*

$$J - RDSkew \to_P \frac{\sum_{t\le T}(\Delta_t Z)^3}{(\int_0^T \sigma_t^2 dt + \sum_{t\le T}(\Delta_t Z)^2)^{3/2}}, \quad (7)$$

$$J - RDKurt \to_P \frac{\sum_{t\le T}(\Delta_t Z)^4}{(\int_0^T \sigma_t^2 dt + \sum_{t\le T}(\Delta_t Z)^2)^2} \quad (8)$$

We set the limits as the measure of skewness and kurtosis under the high frequency situation with the jump-diffusion driven underlying. We aim to estimate this quantity (note that both of them are random variables) using noisy high frequency data.

2.3. Estimation with microstructure noise

The above proposed estimators can not typically be used to the financial data with higher frequency, say tick-by-tick, or even one-minute, because high frequency data inevitably is contaminated by microstructure noise, which is usually induced by bid-ask bounce, rounding error, etc. More specifically, we observe the noisy process $M_i = Z_i + \epsilon_i$, rather than the latent process Z, at points t_i, $0 \le i \le n$.

To deal with the noisy data, existing approaches including two-time scales method proposed by Zhang et al. (2005), preaveraging method studied by Jacod et al. (2009) and further developed by Jacod et al. (2010), and realized kernel method suggested by Barndorff-Nielsen et al. (2008). In this paper, we use the preaveraging approach as described in Jacod et al. (2010). Here we briefly describe the procedure, although it is well-known already. Suppose that the latent process or the log price of a security is Z, but we only can observe $M = Z + \epsilon$, where $\epsilon \perp Z$. As shown in Zhang et al. (2005), if the noise left untreated, will plays dominant role in the estimator. To reduce the effect of microstructure noise, a block with length k_n is constructed. For example, in ith block, we use $\Delta M_i, \Delta M_{i+1}, \cdots, \Delta M_{i+k_n-1}$ to construct a sequence of preaveraging returns for $1 \le i \le n - k_n$:

$$\Delta^n_{i,k_n} M(g) = \sum_{j=1}^{k_n} g(\frac{j}{k_n})\Delta M_{i+j}, \quad \Delta^n_{i,k_n} \bar{M}(g) = \sum_{j=1}^{k_n} \Big((g(\frac{j}{k_n}) - g(\frac{j-1}{k_n}))\Delta M_{i+j}\Big)^2, \tag{9}$$

with a non-negative piece-wise differentiable Lipschitz function g, satisfying $g(x) = 0$ when $x \notin (0,1)$ and $\bar{g}(p) = \int_0^1 g^p(x)dx > 0$. Consequently, the effect of microstructure noise is significantly reduced to a matched order of the latent increment.

Our estimator is constructed basing on preaveraging returns. We collect all the preaveraging returns by the following way:

$$U_n(M,p) = \sum_{i=1}^{n-k_n} (\Delta^n_{i,k_n} M(g))^p, \quad \bar{U}_n(M) = \sum_{i=1}^{n-k_n} \Delta^n_{i,k_n} \bar{M}(g) \tag{10}$$

We have the following asymptotics for the above statistic.

THEOREM 4. *$Z = Y$ is defined as in (1), we have*

$$\frac{1}{k_n} U_n(M,p) \to_P \bar{g}(p) \sum_{s \le T} (\Delta_s Y)^p \quad \textit{For all } p > 2, \tag{11}$$

$$\frac{1}{k_n} U_n(M,2) - \frac{1}{2k_n} \bar{U}_n(M) \to_P \bar{g}(2)[Y,Y]_T = \bar{g}(2)(\int_0^T \sigma_s^2 ds + \sum_{s \le T} (\Delta_s Y)^2). \tag{12}$$

Inspired by above results, using the Slustky's theorem, we obtain a consistent estimator of skewness and kurtosis which we set in the earlier section.

THEOREM 5. *$Z = Y$ is defined as in (1), we have*

$$\frac{(\bar{g}(2))^{3/2}}{\bar{g}(3)} \frac{\sqrt{k_n} U_n(M,3)}{(U_n(M,2) - \bar{U}_n(M)/2)^{3/2}} \to_P \frac{\sum_{s \le T} (\Delta_s Y)^3}{(\int_0^T \sigma_s^2 ds + \sum_{s \le T} (\Delta_s Y)^2)^{3/2}}, \tag{13}$$

$$\frac{(\bar{g}(2))^2}{\bar{g}(4)} \frac{k_n U_n(M,4)}{(U_n(M,2) - \bar{U}_n(M)/2)^2} \to_P \frac{\sum_{s \le T} (\Delta_s Y)^4}{(\int_0^T \sigma_s^2 ds + \sum_{s \le T} (\Delta_s Y)^2)^2}. \tag{14}$$

References

Amaya, D., Christoffersen, P., Jacobs, K., Vasquez, A., 2011. Does realized skewness and kurtosis predict the cross-section of equity returns? Working paper.

Ang, A., Hodrick, R.J., Xing, Y., Zhang, X., 2006. The cross-section of volatility and expected returns. The Journal of Finance 61, 259–299.

Bakshi, G., Kapadia, N., Madan, D., . Stock return characteristics, skew laws, and the differential pricing of individual equity options. Review of Financial Studies 16, 101–143.

Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A., Shephard, N., 2008. Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise. Econometrica 76, 1481–1536.

Barro, R.J., 2009. Rare disasters, asset prices, and welfare costs. American Economic Review 99, 243–64.

Chen, J., Ang, A., Xing, Y., 2006. Downside risk. Review of Financial Studies 19, 1191–1239.

Das, S.R., Sundaram, R.K., 1997. Taming the skew: Higher-order moments in modeling asset price processes in finance. NBER working paper No.5967.

Harvey, C.R., Siddique, A., 2000. Conditional skewness in asset pricing tests. The Journal of Finance 55, 1263–1295.

Jacod, J., Li, Y., Mykland, P.A., Podolskij, M., Vetter, M., 2009. Microstructure noise in the continuous case: the pre-averaging approach. Stochastic Processes and Their Applications 119, 2249–2276.

Jacod, J., Podolskij, M., Vetter, M., 2010. Limit theorems for moving averages of discretized processed plus noise. Ann. Statist. 38, 1478–1545.

Kraus, A., Litzenberger, R.H., 1976. Skewness preference and the valuation of risk assets. Journal of Finance 31, 1085–1100.

Neuberger, A., 2012. Realized skewness. Review of Financial Studies, forthcoming.

Xing, Y., Zhang, X., Zhao, R., 2010. What does the individual option volatility smirk tell us about future equity returns. Journal of Financial and Quantitative Analysis 45, 641–662.

Zhang, L., Mykland, P., Aït-Sahalia, Y., 2005. A tale of two time scales: Determining integrated volatility with noisy high-frequency data. Journal of the American Statistical Association 100, 1394–1411.