Asymptotic expansions for moments of skew normal extremes

Xin Liao[1,3], Zuoxiang Peng[1], and Saralees Nadarajah[2]
[1]School of Mathematics and Statistics, Southwest University, Chongqing, China
[2]School of Mathematics, University of Manchester, Manchester, United Kingdom
[3]Correspongding author: Xin Liao, e-mail: liaoxin2010@163.com

Abstract

Skew normal distribution, an extension of normal distribution, has been widely used in applied statistics, engineer, meteorology and financial time series modeling. We have considered the Mills type inequalities and Mills type ratios of skew normal distribution which are applied to establish the asymptotic expansions of distributions of extremes. In this short note, we focus on the limiting behaviors of moments for normalized partial maxima of skew-normal samples. Under optimal norming constants, asymptotic expansions for moments of maxima of skew-normal samples are derived. These expansions are used to deduce convergence rates of moments of the normalized maxima to the moments of the corresponding extreme value distribution.

Keywords: Extreme value distribution, Maximum, Rate of convergence, Skew normal distribution.

1. Introduction

The biggest weakness of the normal distribution is its inability to model skewed data. This has led to several skewed extensions of the normal distribution. The most popular and the most studied of these extensions is the skew normal distribution due to Azzalini (1985). A random variable X is said to have a standard skew normal distribution with shape parameter $\lambda \in R$ (written as $X \sim \mathbb{SN}(\lambda)$) if its probability density function (pdf) is

$$f_\lambda(x) = 2\phi(x)\Phi(\lambda x), \quad -\infty < x < +\infty, \tag{1}$$

where $\phi(\cdot)$ denotes the standard normal pdf and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function (cdf). It is known that $\mathbb{SN}(0)$ is a standard normal random variable. Let $F_\lambda(\cdot)$ denote the cdf corresponding to (1).

The skew normal distribution has received more applications than any other extension of the normal distribution. Its applications are too many to list. Some applications of the skew normal distribution that have appeared in the past year alone include: the distribution of threshold voltage degradation in nanoscale transistors by using reaction-diffusion and percolation theory (Islam and Alam, 2011); population structure of Schima superba in Qingliangfeng National Nature Reserve (Liu et al., 2011); rain height models to predict fading due to wet snow on terrestrial links (Paulson and Al-Mreri, 2011); modeling of seasonal rainfall in Africa (Siebert and Ward, 2011); modeling of HIV viral loads (Bandyopadhyay et al., 2012); multisite flooding hazard assessment in the Upper Mississippi River (Ghizzoni et al., 2012); modeling of diabetic macular Edema data (Mansourian et al., 2012); risks of macroeconomic forecasts (Pinheiro and Esteves, 2012); modeling of current account balance data (Saez et al., 2012); automated neonatal EEG classification (Temko et al., 2012).

The aim of this short note is to establish asymptotic expansions for moments of M_n, the maximum from independent and identical random variables following the skew normal distribution. The asymptotes of the moments of M_n, the maximum of independent and identical random variables from any given cdf F, have been of considerable interest. McCord (1964) considered convergence of normalized moments of M_n when the random variables belonged to three different classes. Pickands (1968) showed that moments of normalized extremes converge to corresponding moments of the extreme value distribution provided that the moments are finite for sufficiently large n and F is in the domain of attraction of an extreme value distribution. Hall (1979) derived optimal convergence rates for the cdf of M_n when F is a normal cdf. Nair (1981) derived asymptotic expansions for the moments of M_n when F is a normal cdf. For other work, see Ramachandran (1984), Hill and Spruill (1994), Hüsler et al. (2003) and Withers and Nadarajah (2011).
For $r > 0$, let $m_{r,\lambda}(n)$ and m_r denote the rth moments of $F_\lambda^n(a_n x + b_n)$ and $\Lambda(x) = \exp\{-\exp(-x)\}$, respectively; i.e.,

$$m_{r,\lambda}(n) = \int_{-\infty}^{+\infty} x^r dF_\lambda^n\left(a_n x + b_n\right) \qquad m_r = \int_{-\infty}^{+\infty} x^r d\Lambda(x).$$

The asymptotic expansion for the moments of M_n given by Nair (1981) when F is a normal cdf is:

$$\begin{aligned} & \lim_{n\to\infty} b_n^2\left[b_n^2\left(m_{r,0}(n) - m_r\right) + 2^{-1} r\left(m_{r+1} + 2m_r\right)\right] \\ = \; & 8^{-1} r\left[(r+3)m_{r+2} + (4r+12)m_{r+1} + (4r+20)m_r\right], \end{aligned}$$

where $1 - F_0(b_n) = n^{-1}$ and $a_n = b_n^{-1}$.
To establish asymptotic expansions for moments of skew normal extremes, we cite the following result due to Liao et al. (2012):

$$b_n^2\left[b_n^2\left(F_\lambda^n\left(a_n x + b_n\right) - \Lambda(x)\right) - \kappa(x)\Lambda(x)\right] \to \left(\omega(x) + \frac{\kappa^2(x)}{2}\right)\Lambda(x)$$

as $n \to \infty$, where the norming constants a_n, b_n, and the functions $\kappa(x)$, $\omega(x)$ are defined as

(i). for $\lambda \geq 0$, a_n, b_n are given by

$$1 - F_\lambda\left(b_n\right) = n^{-1}, \qquad a_n = b_n^{-1}, \tag{2}$$

and $\kappa(x)$, $\omega(x)$ are defined by

$$\begin{aligned} \kappa(x) &= \left(x^2/2 + x\right)e^{-x}, \\ \omega(x) &= -\left(x^4/8 + x^3/2 + x^2 + 2x\right)e^{-x}. \end{aligned}$$

(ii). for $\lambda < 0$, a_n, b_n are given by

$$1 - F_\lambda\left(b_n\right) = n^{-1}, \qquad a_n = \left(\left(1+\lambda^2\right)b_n\right)^{-1}, \tag{3}$$

and $\kappa(x)$, $\omega(x)$ are defined by

$$\begin{aligned} \kappa(x) &= \left(1+\lambda^2\right)^{-1}\left(x^2/2 + 2x\right)e^{-x}, \\ \omega(x) &= -\lambda^{-2}\left(1+\lambda^2\right)^{-2}\left(\lambda^2 x^4/8 + \lambda^2 x^3 + 3\lambda^2 x^2 + 2\left(1+3\lambda^2\right)x\right)e^{-x}. \end{aligned}$$

The contents of this short note are organized as follows. Section 2 gives the main result on asymptotic expansions for moments of partial maxima of $\mathbb{SN}(\lambda)$ random variables. Some auxiliary lemmas needed to prove the main result are given in Section 3. The proof of the main result is given in Section 4. Throughout, we assume that the parameter $\lambda \neq 0$.

2. Main results

Our main result provides asymptotic expansions for moments of skew normal extremes. These expansions differ according to the sign of λ.

Theorem 1. *Let* $m_r(n) := m_{r,\lambda}(n) = \int_{-\infty}^{+\infty} x^r dF_\lambda^n(a_n x + b_n)$ *and* $m_r = \int_{-\infty}^{+\infty} x^r d\Lambda(x)$, *where* a_n *and* b_n *are given by* (2) *and* (3). *Then,*

(i). for $\lambda > 0$, *we have*

$$b_n^2\Big[b_n^2\left(m_r(n) - m_r\right) + 2^{-1}r\left(m_{r+1} + 2m_r\right)\Big] \to 8^{-1}r\Big[(r+3)m_{r+2} + (4r+12)m_{r+1} + (4r+20)m_r\Big] \quad (4)$$

as $n \to \infty$.

(ii). for $\lambda < 0$, *we have*

$$b_n^2\Big[b_n^2\left(m_r(n) - m_r\right) + 2^{-1}\left(1+\lambda^2\right)^{-1} r\left(m_{r+1} + 4m_r\right)\Big] \to 8^{-1}\lambda^{-2}\left(1+\lambda^2\right)^{-2} r\Big[\lambda^2(r+3)m_{r+2} + 8\lambda^2(r+3)m_{r+1} + 16\left(\lambda^2 r + 4\lambda^2 + 1\right)m_r\Big] \quad (5)$$

as $n \to \infty$.

3. Auxiliary lemmas

To prove the main theorem, we need some auxiliary results. All proofs are omitted here.

Lemma 1. *For* $0 < d < 1$ *and* $i, j \geq 0$, *we have*

$$\lim_{n\to\infty} \int_{-\infty}^{-d\log b_n} b_n^i |x|^j \Lambda(x) dx = 0.$$

Lemma 2. *For* $0 < d < 1$ *and* $i, j \geq 0$, *we have*

$$\lim_{n\to\infty} \int_{-\infty}^{-d\log b_n} b_n^i |x|^j F_\lambda^n\left(a_n x + b_n\right) dx = 0.$$

Lemma 3. *Let* $h_\lambda(b_n, x) = n\log F_\lambda(a_n x + b_n) + e^{-x}$, *where* a_n *and* b_n *are given by* (2) *and* (3). *For sufficiently large* n, *we have*

$$\left|h_\lambda\left(b_n, x\right)\right| < 3$$

uniformly for all $x > -d\log b_n$ *with* $0 < d < 1$.

Lemma 4. *For large* n *and for* $x > -d\log b_n$, $x^r b_n^2\left(b_n^2\left(F_\lambda^n\left(a_n x + b_n\right) - \Lambda(x)\right) - \kappa(x)\Lambda(x)\right)$ *is bounded by integrable functions independent of* n, *where* $r > 0$ *and* $0 < d < 1$.

4. Proof of Theorem 1

First note by Proposition 2.1(iii) of Resnick (1987),

$$\lim_{n\to\infty} m_r(n) = \lim_{n\to\infty} \mathbb{E}\left(\frac{M_n - b_n}{a_n}\right)^r = m_r = \int_{-\infty}^{\infty} x^r d\Lambda(x) = (-1)^r \Gamma^{(r)}(1)$$

since $\int_{-\infty}^0 |x|^r f_\lambda(x)dx \le 2\int_{-\infty}^0 |x|^r \phi(x)dx < \infty$ for all positive integer r, where $\Gamma^{(r)}(1)$ is the rth derivative of the gamma function at $x = 1$. So, for large n, $m_r(n)$ is finite and

$$\begin{aligned} m_r(n) - m_r &= \int_{-\infty}^{\infty} x^r \Big(F_\lambda^n(a_n x + b_n) - \Lambda(x)\Big)' dx \\ &= \int_{-\infty}^{\infty} x^r d\Big(F_\lambda^n(a_n x + b_n) - \Lambda(x)\Big). \end{aligned}$$

By Proposition 2 of Liao et al. (2012), we have

$$1 - F_\lambda(x) = c(x)\exp\left(-\int_1^x \frac{g(t)}{f(t)}dt\right),$$

where $c(x) \to c > 0$ and $g(x) \to 1$ as $x \to \infty$, and auxiliary function $f > 0$ on $(1, \infty)$ that is absolutely continuous with $\lim_{x\to\infty} f'(x) = 0$. Recall that

$$1 - F_\lambda(b_n) = n^{-1}, \qquad a_n = f(b_n),$$

cf. (2) and (3) and Proposition 2 of Liao et al. (2012). By arguments similar to Lemma 2.2(a) of Resnick (1987), we have

$$1 - F_\lambda^n(a_n x + b_n) \le (1+\varepsilon)^2(1+\varepsilon x)^{-\varepsilon^{-1}+1} \tag{6}$$

for $x > 0$, arbitrary $\varepsilon > 0$ and large n. Hence, for $0 < \varepsilon < \frac{1}{r+1}$, we have

$$0 \le \limsup_{x\to\infty} x^r(1 - F_\lambda^n(a_n x + b_n)) \le \lim_{x\to\infty} x^r(1+\varepsilon)^2(1+\varepsilon x)^{-\varepsilon^{-1}+1} = 0$$

by (6), implying

$$\lim_{x\to\infty} x^r(1 - F_\lambda^n(a_n x + b_n)) = 0. \tag{7}$$

Noting that $\int_{-\infty}^0 |x|^r f_\lambda(x)dx < \infty$ implies $\lim_{x\to-\infty} |x|^r F_\lambda(x) = 0$, and by C_r-inequality we have

$$0 \le \limsup_{x\to-\infty} |x|^r F_\lambda^n(a_n x + b_n) \le \lim_{y\to-\infty} \frac{2^{r-1}(|y|^r + |b_n|^r)}{a_n^r} F_\lambda(y) = 0,$$

which implies

$$\lim_{x\to-\infty} x^r F_\lambda^n(a_n x + b_n) = 0. \tag{8}$$

Hence by (7) and (8) we have

$$\lim_{x\to\infty} x^r\left(F_\lambda^n(a_n x + b_n) - \Lambda(x)\right) = \lim_{x\to\infty} x^r(1 - \Lambda(x)) - \lim_{x\to\infty} x^r(1 - F_\lambda^n(a_n x + b_n)) = 0 \tag{9}$$

and

$$\lim_{x\to-\infty} x^r\left(F_\lambda^n(a_n x + b_n) - \Lambda(x)\right) = \lim_{x\to-\infty} x^r F_\lambda^n(a_n x + b_n) - \lim_{x\to-\infty} x^r \Lambda(x) = 0. \tag{10}$$

So, by integration by parts, and (9), (10), we have

$$\begin{aligned} m_r(n) - m_r &= \int_{-\infty}^{+\infty} x^r d\left(F_\lambda^n(a_n x + b_n) - \Lambda(x)\right) \\ &= -r\int_{-\infty}^{+\infty} x^{r-1}\left(F_\lambda^n(a_n x + b_n) - \Lambda(x)\right) dx \end{aligned} \tag{11}$$

for large n. Noting that

$$\int_{-\infty}^{+\infty} x^k e^{-2x}\Lambda(x)\,dx = \int_{-\infty}^{+\infty} x^k e^{-x} d\Lambda(x) = -km_{k-1} + m_k,$$

and by Theorem 2 in Liao et al. (2012), (11), Lemmas 1-4 and the dominated convergence theorem, we have

$$\begin{aligned} & b_n^2\left[b_n^2\Big(m_r(n) - m_r\Big) + 2^{-1}r\left(m_{r+1} + 2m_r\right)\right] \\ = & -r\int_{-\infty}^{+\infty} b_n^2\left[b_n^2 x^{r-1}\Big(F_\lambda^n(a_n x + b_n) - \Lambda(x)\Big) - x^{r-1}\kappa(x)\Lambda(x)\right]dx \\ = & -r\int_{-\infty}^{-d\log b_n} b_n^2\left[b_n^2 x^{r-1}\Big(F_\lambda^n(a_n x + b_n) - \Lambda(x)\Big) - x^{r-1}\kappa(x)\Lambda(x)\right]dx \\ & -r\int_{-d\log b_n}^{+\infty} b_n^2\left[b_n^2 x^{r-1}\Big(F_\lambda^n(a_n x + b_n) - \Lambda(x)\Big) - x^{r-1}\kappa(x)\Lambda(x)\right]dx \\ \to & -r\int_{-\infty}^{+\infty}\left(\omega(x) + \frac{(\kappa(x))^2}{2}\right) x^{r-1}\Lambda(x)\,dx \\ = & \; 8^{-1}r\Big[(r+3)\,m_{r+2} + (4r+12)\,m_{r+1} + (4r+20)\,m_r\Big] \end{aligned}$$

as $n \to \infty$ for $\lambda > 0$, which is (4).
Similarly, we can derive (5) for $\lambda < 0$. The proof is complete. □

Acknowledgements This work was supported by the National Natural Science Foundation of China grant no.11171275, the Natural Science Foundation Project of CQ no. cstc2012jjA00029 and the SWU grant for Statistics Ph.D.

References

Azzalini, A. (1985) "A class of distributions which includes the normal ones," *Scandinavian Journal of Statistics*, 12, 171-178.
Bandyopadhyay, D., Lachos, V. H., Castro, L. M. and Dey, D. K. (2012) "Skew-normal/independent linear mixed models for censored responses with applications to HIV viral loads," *Biometrical Journal*, 54, 405-425.
Ghizzoni, T., Roth, G. and Rudari, R. (2012) "Multisite flooding hazard assessment in the Upper Mississippi River," *Journal of Hydrology*, 412, 101-113.
Hall, P. (1979) "On the rate of convergence of normal extremes," *Journal of Applied Probability*, 16, 433-439.
Hill, T. and Spruill, M. (1994) "On the relationship between convergence in distribution and convergence of expected extremes," *Proceedings of the American Mathematical Society*, **121**, 1235-1243.

Hüsler, J., Piterbarg, V. and Seleznjev, O. (2003) "On convergence of the uniform norms for Gaussian processes and linear approximation problems," *Annals of Applied Probability*, 13, 1615-1653.

Islam, A. E. and Alam, M. A. (2011) "Analyzing the distribution of threshold voltage degradation in nanoscale transistors by using reaction-diffusion and percolation theory," *Journal of Computational Electronics*, 10, 341-351.

Liao, X., Peng, Z., Nadarajah, S. and Wang, X. (2012) "Rates of convergence of extremes from skew normal samples," Available on-line at http://arxiv.org/abs/1212.1004.

Liu, X. -H., Li, Y. -G., Shen, G. -C., Weng, D. -M., Zhang, H. -W. and Zhang, F. -G. (2011) "Population structure of Schima superba in Qingliangfeng National Nature Reserve," *Forest Research*, 24, 28-32.

Mansourian, M., Kazemnejad, A., Kazemi, I., Zayeri, F. and Soheilian, M. (2012) "Bayesian analysis of longitudinal ordered data with flexible random effects using McMC: application to diabetic macular Edema data," *Journal of Applied Statistics*, 39, 1087-1100.

McCord, J. R. (1964) "On asymptotic moments of extreme statistics," *Annals of Mathematical Statistics*, 64, 1738-1745.

Nair, K. A. (1981) "Asymptotic distribution and moments of normal extremes," *Annals of Probability*, 9, 150-153.

Paulson, K. and Al-Mreri, A. (2011) "A rain height model to predict fading due to wet snow on terrestrial links," *Radio Science*, 46, Article Number: RS4010.

Pickands, J. (1968) "Moment convergence of sample extremes," *Annals of Mathematical Statistics*, 39, 881-889.

Pinheiro, M. and Esteves, P. S. (2012) "On the uncertainty and risks of macroeconomic forecasts: combining judgements with sample and model information," *Empirical Economics*, 42, 639-665.

Ramachandran, G. (1984) "Approximate values for the moments of extreme order statistics in large samples," *Statistical Extremes and Applications* (Vimeiro, 1983), 563-578, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 131, Reidel, Dordrecht.

Resnick, S. I. (1987) *Extreme Values, Regular Variation, and Point Processes*, Springer Verlag, New York.

Saez, A. J., Prieto, F. and Sarabia, J. M. (2012) "A two-tail version of the PPS distribution with application to current account balance data," *Physica A—Statistical Mechanics and Its Applications*, 391, 5160-5171.

Siebert, A. B. and Ward, M. N. (2011) "Future occurrence of threshold-crossing seasonal rainfall totals: Methodology and application to sites in Africa," *Journal of Applied Meteorology and Climatology*, 50, 560-578.

Temko, A., Stevenson, N., Marnane, W., Boylan, G. and Lightbody, G. (2012) "Inclusion of temporal priors for automated neonatal EEG classification," *Journal of Neural Engineering*, 9, Article Number: 046002.

Withers, C. S. and Nadarajah, S. (2011) "Expansions for quantiles and multivariate moments of extremes for distributions of Pareto type," *Sankhyā*, A, 2, 202-217.