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Berry-Esséen bounds and almost sure CLT for the quadratic variation of the bifractional Brownian motion

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Abstract

Let B be a bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1]$. For any $n \geq 1$, set $Z_n = \sum_{i=0}^{n-1} \left[n^{2HK}(B_{(i+1)/n} - B_{i/n})^2 - \mathbb{E}\left((B_{i+1} - B_i)^2\right) \right]$. We use the Malliavin calculus and the so-called Stein's method on Wiener chaos introduced by Nourdin and Peccati [11] to derive, in the case when $0 < HK \leq 3/4$, Berry-Esséen-type bounds for the Kolmogorov distance between the law of the correct renormalization V_n of Z_n and the standard normal law. Finally, we study almost sure central limit theorems for the sequence V_n.

Key words : Kolmogorov distance; Central limit theorem; Almost sure central limit theorem; Bifractional Brownian motion; Multiple stochastic integrals; Quadratic variation.

1 Introduction

Let $B = (B_t, t \geq 0)$ be a bifractional Brownian motion (bifBm) with parameters $H \in (0,1)$ and $K \in (0,1]$, defined on some probability space $(\Omega, \mathcal{F}, P)$. (Here, and everywhere else, we do assume that $\mathcal{F}$ is the sigma-field generated by B.) This means that B is a centered Gaussian process with the covariance function $E[B_sB_t] = R_{H,K}(s,t)$, where

$$R_{H,K}(s,t) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right). \tag{1.1}$$

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The case $K = 1$ corresponds to the fractional Brownian motion (fBm) with Hurst parameter H. The process B has no stationary increments, but it has the quasi-helix property (in the sense of J.P. Kahane),

$$2^{-K}|t-s|^{2HK} \leq \mathbb{E}\left(|B_t - B_s|^2\right) \leq 2^{1-K}|t-s|^{2HK}, \tag{1.2}$$

so B has $\gamma-$Hölder continuous paths for any $\gamma \in (0, HK)$ thanks to the Kolmogorov-Centsov theorem, and it is a self-similar process, that is, for any constant $a > 0$, the processes $(B_{at},\ t \geq 0)$ and $(a^{HK}B_t,\ t \geq 0)$ have the same distribution. The bifBm B can be extended for $1 < K < 2$ with $H \in (0,1)$ and $HK \in (0,1)$ (see [1]). We refer to [8, 13, 6, 9] for further details on the subject.

An example of interesting problem related to B is the study of the asymptotic behavior of the quadratic variation of B on $[0,1]$ defined as

$$Z_n = \sum_{i=0}^{n-1}\left[n^{2HK}(B_{(i+1)/n} - B_{i/n})^2 - \mathbb{E}\left((B_{i+1} - B_i)^2\right)\right], \quad n \geq 1.$$

Let us consider the correct renormalization V_n of Z_n given as,

$$V_n = \frac{Z_n}{\sqrt{Var(Z_n)}}. \tag{1.3}$$

Recall that, if Y, Z are two real-valued random variables, then the Kolmogorov distance between the law of Y and the law of Z is given by

$$d_{\mathrm{Kol}}(Y,Z) = \sup_{-\infty<z<\infty}|P(Y \leq z) - P(Z \leq z)|.$$

In the particular case of the fBm (that is when $K = 1$), and thanks to the seminal works of Breuer and Major [4], Dobrushin and Major [5], Giraitis and Surgailis [7] and Taqqu [14], it is well-known that we have, as $n \to \infty$:

- If $0 < H < \frac{3}{4}$ then
$$\frac{V_n}{\sigma_H\sqrt{n}} \xrightarrow{\text{law}} \mathcal{N}(0,1).$$
- If $H = \frac{3}{4}$ then
$$\frac{V_n}{\sigma_H\sqrt{n\log(n)}} \xrightarrow{\text{law}} \mathcal{N}(0,1).$$
- If $H > \frac{3}{4}$ then
$$\frac{V_n}{n^{2H-1}} \xrightarrow{\text{law}} Z \sim \text{“Hermite random variable”}\ .$$

Here, $\sigma_H > 0$ denotes an (explicit) constant depending only on H. Moreover, explicit bounds for the Kolmogorov distance between the law of V_n and the standard normal law are obtained by [11, Theorem 4.1], [3, Theorem 1.2] and [10, Theorem 5.6]. The following facts happen: For some constant c_H depending only on H, we have:

$$d_{Kol}\left(V_n, \mathcal{N}(0,1)\right) \leq c_H \times \begin{cases} \frac{1}{\sqrt{n}} & \text{if } H \in \left(0, \frac{5}{8}\right) \\ \frac{(\log n)^{3/2}}{\sqrt{n}} & \text{if } H = \frac{5}{8} \\ n^{4H-3} & \text{if } H \in \left(\frac{5}{8}, \frac{3}{4}\right) \\ \frac{1}{\sqrt{\log n}} & \text{if } H = \frac{3}{4} \end{cases}$$

On other hand, Bercu et al. [2] proved the almost sure central limit theorem (ASCLT) for V_n. Recently, Tudor [15] studied the subfractional Brownian motion case.

Let us now describe the results we prove in the present paper. First, in Theorem 3.1 we use the Malliavin calculus and Stein method, in the case when $HK \in (0, \frac{3}{4}]$, to derive explicit bounds for the Kolmogorov distance between the law of V_n and the standard normal law. Precisely, three cases are considered according to the value of HK:

$$d_{Kol}\left(V_n, \mathcal{N}(0,1)\right) \leq c_{H,K} \times \begin{cases} n^{-\frac{1}{2}} & \text{if } HK \in \left(0, \frac{1}{2}\right] \\ n^{2HK-\frac{3}{2}} & \text{if } HK \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \frac{1}{\sqrt{\log n}} & \text{if } HK = \frac{3}{4} \end{cases}$$

where $c_{H,K}$ is a constant depending only on H and K. In Theorem 4.1, we prove almost sure central limit theorem for V_n.

The rest of the paper is organized as follows. Section 2 deals with preliminaries concerning Malliavin calculus, Stein's method and related topics needed throughout the paper. Section 3 and 4 contain our main results, concerning Berry-Essén bounds and ASCLT for the quadratic variation of the bifractional Brownian motion.

2 Preliminaries

In this section, we briefly recall some basic facts concerning Gaussian analysis and Malliavin calculus that are used in this paper; we refer to [12] for further details. Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q \geq 1$, we denote by $\mathfrak{H}^{\otimes q}$ (resp. $\mathfrak{H}^{\odot q}$) the qth tensor product (resp. qth symmetric tensor product) of $\mathfrak{H}$. We write $X = \{X(h), h \in \mathfrak{H}\}$ to indicate a centered isonormal Gaussian process on $\mathfrak{H}$. This means that X is a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, P)$ and such that $E[X(g)X(h)] = \langle g, h\rangle_{\mathfrak{H}}$ for every $g, h \in \mathfrak{H}$. (Here, and everywhere else, we do assume that $\mathcal{F}$ is the sigma-field generated by X.)

For every $q \geq 1$, let $\mathcal{H}_q$ be the q^{th} Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the q^{th} Hermite polynomial defined as $H_q(x) = (-1)^q \, e^{\frac{x^2}{2}} \, \frac{d^q}{dx^q} \left(e^{-\frac{x^2}{2}} \right)$. The mapping $I_q(h^{\otimes q}) = H_q(X(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\| \cdot \|_{\mathfrak{H}^{\odot q}} = \sqrt{q!} \, \| \cdot \|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$. Specifically, for all $f, g \in \mathfrak{H}^{\odot q}$ and $q \geq 1$, one has

$$E\big[I_q(f) I_q(g)\big] = q! \langle f, g \rangle_{\mathcal{H}^{\otimes q}} \tag{2.4}$$

On the other hand, it is well-known that any random variable Z belonging to $L^2(\Omega)$ admits the following chaotic expansion:

$$Z = E[Z] + \sum_{q=1}^{\infty} I_q(f_q) \tag{2.5}$$

where the series converges in $L^2(\Omega)$ and the kernels f_q, belonging to $\mathfrak{H}^{\odot q}$, are uniquely determined by Z.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the rth contraction of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1=1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

In particular, note that $f \otimes_0 g = f \otimes g$ and when $p = q$, that $f \otimes_p g = \langle f, g \rangle_{\mathfrak{H}^{\otimes p}}$. Since, in general, the contraction $f \otimes_r g$ is not necessarily symmetric, we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. When $f \in \mathfrak{H}^{\odot q}$, we write $I_q(f)$ to indicate its qth multiple integral with respect to X. The following formula is useful to compute the product of such multiple integrals: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \tag{2.6}$$

Let $\mathcal{S}$ be the set of all smooth cylindrical random variables, that is, which can be expressed as $F = f(X(\phi_1), \ldots, X(\phi_n))$ where $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function such that f and all its derivatives have at most polynomial growth, and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the square integrable $\mathfrak{H}$-valued random variable defined as

$$DF \; = \; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (X(\phi_1), \ldots, X(\phi_n)) \phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 \; = \; E[F^2] + E\big[\|DF\|_{\mathfrak{H}}^2\big].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}_b^1$ and if $(F_i)_{i=1,\dots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, G_n) \in \mathbb{D}^{1,2}$ and we have

$$D\varphi(F_1, \dots, G_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \dots, G_n) DF_i.$$

Recall the following results concerning CLT and ASCLT for multiple stochastic integrals.

Theorem 2.1 (Nourdin-Peccati [11]) *Let $q \geq 2$ be an integer and let $F = I_q(f)$ with $f \in \mathcal{H}^{\odot q}$, then*

$$d_{Kol}(F, N) \leq \sqrt{E\left[\left(1 - \frac{1}{q}\|DF\|_{\mathcal{H}}^2\right)^2\right]}, \tag{2.7}$$

where $N \sim \mathcal{N}(0,1)$.

Theorem 2.2 (Bercu et al. [2]) *Let $q \geq 2$ be an integer, and let $\{G_n\}_{n\geq 1}$ be a sequence of the form $G_n = I_q(f_n)$, with $f_n \in \mathcal{H}^{\odot q}$. Assume that $E[G_n^2] = q!\|f_n\|_{\mathfrak{H}^{\otimes q}}^2 = 1$ for all n, and that $G_n \stackrel{\text{law}}{\longrightarrow} N \sim \mathcal{N}(0,1)$ as $n \to \infty$. If the two following conditions are satisfied*

1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(q-r)}} < \infty$ *for avery* $1 \leq r \leq q-1$,

2) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{|\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}}|}{kl} < \infty$,

then $\{G_n\}_{n\geq 1}$ satisfies an ASCLT. In other words, almost surely, for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log(n)} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \to \mathbb{E}\varphi(N), \quad \text{as } n \to \infty.$$

From now, assume on one hand that $X = B$ is a bifBm with parameters $H \in (0,1)$ and $K \in (0,1]$ and on the other hand $\mathfrak{H}$ is a real separable Hilbert space defined as follows: (i) denote by $\mathcal{E}$ the set of all $\mathbb{R}$-valued step functions on $[0,\infty)$, (ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathfrak{H}} = R_{H,K}(s,t) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right).$$

In particular, one has that $B_t = B(1_{[0,t]})$.

3 Berry-Esséen bounds in the CLT for the quadratic variation of the bifBm

In this section, we prove that for every $HK \in \left(0, \frac{3}{4}\right]$ a Central Limit Theorem holds, where V_n was defined in (1.3). Using the Stein's method we also derive the Berry-Esséen bounds for this convergence.

3.1 General setup

Let us define

$$\theta(i,j) = 2^{-K}\big(\gamma(i,j) + \rho(i-j)\big), \quad i,j \in \mathbb{N}$$

where

$$\begin{aligned}\gamma(i,j) &= \left((i+1)^{2H} + (j+1)^{2H}\right)^K - \left(i^{2H} + (j+1)^{2H}\right)^K - \left((i+1)^{2H} + j^{2H}\right)^K \\ &\quad + \left(i^{2H} + j^{2H}\right)^K, \end{aligned} \tag{3.1}$$

and

$$\rho(r) = |r+1|^{2HK} + |r-1|^{2HK} - 2|r|^{2HK}, \quad r \in \mathbb{Z}. \tag{3.2}$$

Observe that the function γ is symmetric, $\rho(0) = 2$, $\rho(x) = \rho(-x)$ and ρ behaves asymptotically as

$$\rho(r) = 2HK(2HK-1)|r|^{2HK-2}, \quad |r| \to \infty. \tag{3.3}$$

In particular, $\sum_{r\in\mathbb{Z}} \rho^2(r) < \infty$ if, and only if, $HK \in (0, \frac{3}{4})$.
We will use the notation

$$\delta_{k/n} = 1_{[k/n,(k+1)/n]} \quad \text{and} \quad \sigma = \sqrt{\frac{1}{8}\sum_{r\in\mathbb{Z}} \rho^2(r)}. \tag{3.4}$$

Using self-similarity property of B and (1.1) we deduce that

$$\begin{aligned} n^{2HK}\langle \delta_{i/n}, \delta_{j/n}\rangle_{\mathfrak{H}} &= n^{2HK}\mathbb{E}\left(\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right)\left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}}\right)\right) \\ &= \mathbb{E}\left((B_{i+1} - B_i)(B_{j+1} - B_j)\right) \\ &= \theta(i,j). \end{aligned}$$

Hence, we can write the quadratic variation of B, with respect to a subdivision $\pi_n = \{0 < \frac{1}{n} < \frac{2}{n} < \ldots < 1\}$ of $[0,1]$, as follows

$$\begin{aligned} Z_n &= \sum_{k=0}^{n-1}\left[n^{2HK}\left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\right)^2 - \theta(k,k)\right] \\ &= \sum_{k=0}^{n-1}\left[n^{2HK}\left(I_1(\delta_{k/n})\right)^2 - \theta(k,k)\right] \\ &= I_2\left(\underbrace{n^{2HK}\sum_{k=0}^{n-1}\delta_{k/n}^{\otimes 2}}_{g_n}\right) \\ &= I_2(g_n). \end{aligned} \tag{3.5}$$

Thus, we can also write the correct renormalization V_n, defined in (1.3), of Z_n as follows,

$$V_n = \frac{Z_n}{\sqrt{Var(Z_n)}} = \frac{I_2(g_n)}{\sqrt{Var(Z_n)}}. \tag{3.6}$$

Before computing the Kolmogorov distance, we start with the following results which are used throughout the paper. Here, and everywhere else, the notation $a_n \trianglelefteq b_n$ means that $\sup_{n\geq 1} |a_n|/|b_n| < \infty$.

Lemma 3.1

i) Fixing $y \geq 0$ (resp. $x \geq 0$), the function $x \to \gamma(x,y)$ (resp. $y \to \gamma(x,y)$) defined in (3.1) is increasing for $H \in \left(0, \frac{1}{2}\right]$.

ii) For any $H \in (0,1)$ and $K \in (0,1]$, the function γ is negative and we have for j large

$$\gamma(0,j) \sim c_{H,K} j^{2HK-2}, \tag{3.7}$$
$$\gamma(j,j) \sim c_{H,K} j^{2HK-2}. \tag{3.8}$$

If $j \leq l$ then

$$|\gamma(j,l)| \leq c_{H,K} l^{2HK-2}. \tag{3.9}$$

where $c_{H,K}$ is a constant (explicit) depending only on H and K.

Proof. *i)* We fix $y \geq 0$,

$$\begin{aligned} \frac{\partial\gamma}{\partial x}(x,y) &= 2HK(x+1)^{2H-1}\Big[((x+1)^{2H} + (y+1)^{2H})^{K-1} - ((x+1)^{2H} + y^{2H})^{K-1}\Big] \\ &\quad -2HKx^{2H-1}\Big[(x^{2H} + (y+1)^{2H})^{K-1} - (x^{2H} + y^{2H})^{K-1}\Big] \\ &= 2HK\Big[g(1+x) - g(x)\Big], \end{aligned} \tag{3.10}$$

where

$$g(x) = x^{2H-1}\Big[(x^{2H} + (y+1)^{2H})^{K-1} - (x^{2H} + y^{2H})^{K-1}\Big],$$

If $H \in \left(0, \frac{1}{2}\right]$ and $K \in (0,1]$, then γ is increasing since the function g is increasing on $(0,\infty)$. Indeed,

$$\begin{aligned} g'(x) &= (2H-1)x^{2H-2}\Big[(x^{2H} + (y+1)^{2H})^{K-1} - (x^{2H} + y^{2H})^{K-1}\Big] \\ &\quad +2H(K-1)x^{4H-2}\Big[(x^{2H} + (y+1)^{2H})^{K-2} - (x^{2H} + y^{2H})^{K-2}\Big] \\ &\geq 0. \end{aligned}$$

ii) To show that γ is negative, it suffices to remark the decreasing property of the function $p : x \in [0,\infty) \to (a+x)^K - (b+x)^K$.

By straightforward expansion of function γ, we can easily prove (3.7) and (3.8).
If $H \le \frac{1}{2}$, by the first point $i)$, the function $x \to |\gamma(x,y)|$ is decreasing. Thus, we deduce

$$|\gamma(k,l)| \le |\gamma(0,l)| \sim c_{H,K} l^{2HK-2}.$$

If $H > \frac{1}{2}$, we rewrite γ as $\gamma(k,l) = g_k(1+l) - g_k(l)$ where $g_k(x) := ((k+1)^{2H} + x^{2H})^K - (k^{2H} + x^{2H})^K$. Applying mean value theorem we obtain for some $x_{k,l} \in [l, l+1]$ that

$$\begin{aligned} |\gamma(k,l)| &= 2HKx_{k,l}^{2H-1}\Big[\big(x_{k,l}^{2H} + k^{2H}\big)^{K-1} - \big(x_{k,l}^{2H} + (k+1)^{2H}\big)^{K-1}\Big] \\ &\le 2HK(l+1)^{2H-1}\Big[\big(l^{2H} + k^{2H}\big)^{K-1} - \big(l^{2H} + (k+1)^{2H}\big)^{K-1}\Big]. \end{aligned}$$

Again by mean value theorem on $y \to \big(l^{2H} + y^{2H}\big)^{K-1}$, we have for some $y_{k,l} \in [k, k+1]$

$$\Big[\big(l^{2H} + k^{2H}\big)^{K-1} - \big(l^{2H} + (k+1)^{2H}\big)^{K-1}\Big] = 2H(K-1)y_{k,l}^{2H-1}\Big[l^{2H} + y_{k,l}^{2H}\Big]^{K-2}.$$

Consequently, for $k \le l$,

$$\begin{aligned} |\gamma(k,l)| &\le 4H^2K(1-K)(l+1)^{2H-1}(k+1)^{2H-1}\Big[l^{2H} + k^{2H}\Big]^{K-2} \\ &\le c_{H,K} l^{2HK-2} \end{aligned}$$

and the second point $ii)$ follows. □

Proposition 3.1 *Let Z_n be the sequence defined in (3.5) and let σ be the constant given by (3.4).*

1. *Assume that $0 < HK < \frac{3}{4}$. Then, as $n \to \infty$, it holds*

$$\frac{Var(Z_n)}{4^{2-K} n\sigma^2} \longrightarrow 1. \tag{3.11}$$

2. *Assume that $HK = \frac{3}{4}$. Then, as $n \to \infty$, it holds*

$$\frac{Var(Z_n)}{4^{2-K}\sigma^2 n \log n} \longrightarrow 1. \tag{3.12}$$

Proof. To show (3.11), we write

$$\begin{aligned} \frac{Var(Z_n)}{4^{2-K} n\sigma^2} - 1 &= \frac{n^{-1}}{4^{2-K}\sigma^2}\,\mathbb{E}[I_2^2(g_n)] - 1 = \frac{n^{-1}}{2^{3-2K}\sigma^2}\,\|g_n\|^2_{\mathfrak{H}^{\otimes 2}} - 1 \\ &= \frac{n^{4HK-1}}{2^{3-2K}\sigma^2}\sum_{k,l=0}^{n-1}\langle \delta_{k/n}^{\otimes 2}, \delta_{l/n}^{\otimes 2}\rangle_{\mathfrak{H}^{\otimes 2}} - 1 \\ &= \frac{n^{4HK-1}}{2^{3-2K}\sigma^2}\sum_{k,l=0}^{n-1}\langle \delta_{k/n}, \delta_{l/n}\rangle^2_{\mathfrak{H}} - 1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{n^{-1}}{2^{3-2K}\sigma^2}\sum_{k,l=0}^{n-1}\theta^2(k,l)-1\\
&= \frac{n^{-1}}{8\sigma^2}\sum_{k,l=0}^{n-1}\gamma^2(k,l)+\left(\frac{n^{-1}}{8\sigma^2}\sum_{k,l=0}^{n-1}\rho^2(k-l)-1\right)+\frac{n^{-1}}{4\sigma^2}\sum_{k,l=0}^{n-1}\gamma(k,l)\rho(k-l)\\
&=: J_1(n)+J_2(n)+J_3(n).
\end{aligned}
$$

As in the proof of [11, Theorem 4.1], we have

$$
J_2(n) \longrightarrow 0, \quad \text{as } n\to\infty. \tag{3.13}
$$

On the other hand

$$
\begin{aligned}
J_1(n) &= \frac{n^{-1}}{8\sigma^2}\sum_{k,l=0}^{n-1}\gamma^2(k,l)\\
&= \frac{n^{-1}}{8\sigma^2}\sum_{k=0}^{n-1}\gamma^2(k,k)+\frac{n^{-1}}{4\sigma^2}\sum_{0\le k<l\le n-1}\gamma^2(k,l)\\
&=: J_{1,1}(n)+J_{1,2}(n).
\end{aligned}
$$

By (3.8), the sum

$$
J_{1,1}(n)=\frac{n^{-1}}{8\sigma^2}\sum_{k=0}^{n-1}\gamma^2(k,k),
$$

behaves as $\frac{n^{-1}}{8\sigma^2}\sum_{k=0}^{n-1}k^{4HK-4}$ which goes to zero as $n\to\infty$, because $HK<3/4$.

Thus,

$$
J_{1,1}(n)\longrightarrow 0 \quad \text{as } n\to\infty. \tag{3.14}
$$

Now, we study the convergence of $J_{1,2}(n)$. We first fix two positive constants α and β such that $\alpha+\beta=1$ and $4HK-2<\beta<1$.

We deduce from (3.9) that

$$
\begin{aligned}
J_{1,2}(n)=\frac{n^{-1}}{4\sigma^2}\sum_{0\le k<l\le n-1}\gamma^2(k,l) &\le C_{H,K}\frac{n^{-1}}{4\sigma^2}\sum_{0\le l\le n-1}l^{4HK-3}\\
&\le C_{H,K}\frac{n^{-\alpha}}{4\sigma^2}\sum_{0\le l\le n-1}l^{4HK-3-\beta}\longrightarrow 0, \text{ as } n\to\infty.
\end{aligned}
$$

Hence,

$$
J_{1,2}(n)\longrightarrow 0, \quad \text{as } n\to\infty. \tag{3.15}
$$

Combining (3.14) and (3.15) leads to

$$J_1(n) \longrightarrow 0, \quad \text{as } n \to \infty. \tag{3.16}$$

Finally, from (3.16) and (3.13) together with Cauchy Schwartz inequality

$$\begin{aligned} |J_3(n)| &\leq \frac{n^{-1}}{4\sigma^2} \sum_{k,l=0}^{n-1} |\gamma(k,l)\rho(k-l)| \\ &\leq \left(\frac{n^{-1}}{4\sigma^2} \sum_{k,l=0}^{n-1} \gamma^2(k,l) \right)^{1/2} \left(\frac{n^{-1}}{4\sigma^2} \sum_{k,l=0}^{n-1} \rho^2(k-l) \right)^{1/2} \\ &= 2\sqrt{J_1(n)(J_2(n)+1)} \longrightarrow 0, \text{ as } n \to \infty, \end{aligned} \tag{3.17}$$

and the convergence (3.11) follows.
We prove now (3.12). Following similar argument of the proof of (3.11), we have

$$\begin{aligned} \frac{Var(Z_n)}{4^{2-K}\sigma^2 n \log n} - 1 &= \frac{n^{-1}}{8\sigma^2 \log n} \sum_{k,l=0}^{n-1} \gamma^2(k,l) + \left(\frac{n^{-1}}{8\sigma^2 \log n} \sum_{k,l=0}^{n-1} \rho^2(k-l) - 1 \right) \\ &\quad + \frac{n^{-1}}{4\sigma^2 \log n} \sum_{k,l=0}^{n-1} \gamma(k,l)\rho(k-l) \\ &= \frac{1}{\log n} J_1(n) + \frac{1}{\log n} J_2(n) + \frac{1}{\log n} J_3(n). \end{aligned}$$

From [3, page 490] we have

$$\frac{1}{\log n} J_2(n) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.18}$$

On the other hand, since $HK = \frac{3}{4}$ and the fact that $\log(n) \sim \sum_1^{n-1} \frac{1}{k}$ we deduce easily from (3.16) and (3.17) that

$$\frac{1}{\log n} J_1(n) + \frac{1}{\log n} J_3(n) \underset{n\to\infty}{\longrightarrow} 0 \, .$$

□

3.2 A Berry-Esséen bound for $0 < HK \leq \frac{3}{4}$

Our first main result is summarized in the following Theorem.

Theorem 3.1 *Let $N \sim \mathcal{N}(0,1)$ and let V_n be defined by (3.6). Then V_n converges in distribution to N. In addition, for some constant $c_{H,K}$ depending uniquely on H and K, we* *have: for every* $n \geq 1$,

$$d_{Kol}(V_n, N) \leq c_{H,K} \times \begin{cases} \frac{1}{\sqrt{n}} & \text{if } HK \in \left(0, \frac{1}{2}\right] \\ n^{2HK-\frac{3}{2}} & \text{if } HK \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \frac{1}{\sqrt{\log n}} & \text{if } HK = \frac{3}{4} \end{cases}$$

Proof. From (3.5), we have

$$DZ_n = 2n^{2HK} \sum_{k=0}^{n-1} I_1(\delta_{k/n})\delta_{k/n},$$

then

$$\|DZ_n\|_{\mathfrak{H}}^2 = 4n^{4HK} \sum_{k,l=0}^{n-1} I_1(\delta_{k/n}) I_1(\delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}},$$

by the multiplication formula (2.6), we get

$$\begin{aligned} \|DZ_n\|_{\mathfrak{H}}^2 &= 4n^{4HK} \sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \widetilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} + 4n^{4HK} \sum_{k,l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \\ &= 4n^{4HK} \sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \widetilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} + \mathbb{E}\|DZ_n\|_{\mathfrak{H}}^2. \end{aligned}$$

Combining this with the fact that $\mathbb{E}\|DZ_n\|_{\mathfrak{H}}^2 = 2Var(Z_n)$, we obtain

$$\frac{1}{2}\|DV_n\|_{\mathfrak{H}}^2 - 1 = \frac{2n^{4HK}}{Var(Z_n)} \sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \widetilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}.$$

It follows that

$$\begin{aligned} &\mathbb{E} \left[\left(\frac{1}{2} \|DV_n\|_{\mathfrak{H}}^2 - 1 \right)^2 \right] \\ &= \frac{4n^{8HK}}{Var^2(Z_n)} \mathbb{E} \left[\left(\sum_{k,l=0}^{n-1} I_2(\delta_{k/n} \widetilde{\otimes} \delta_{l/n}) \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \right)^2 \right] \\ &= \frac{8n^{8HK}}{Var^2(Z_n)} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \langle \delta_{i/n} \widetilde{\otimes} \delta_{j/n}, \delta_{k/n} \widetilde{\otimes} \delta_{l/n} \rangle_{\mathfrak{H}^{\otimes 2}}. \qquad (3.19) \\ &= \frac{8n^2}{Var^2(Z_n)} A(n) \qquad (3.20) \end{aligned}$$

where

$$
\begin{aligned}
A(n) &= n^{8HK-2} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \ \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \ \langle \delta_{i/n} \widetilde{\otimes} \delta_{j/n}, \delta_{k/n} \widetilde{\otimes} \delta_{l/n} \rangle_{\mathfrak{H}^{\otimes 2}} \\
&= \frac{n^{8HK-2}}{2} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \ \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \ \Big(\langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \ \langle \delta_{j/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \\
&\qquad + \langle \delta_{i/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \langle \delta_{j/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \Big) \\
&= n^{8HK-2} \sum_{i,j,k,l=0}^{n-1} \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \ \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \ \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \ \langle \delta_{j/n}, \delta_{l/n} \rangle_{\mathfrak{H}}
\end{aligned}
$$

Hence, using that fact that for every $a, b \in \mathbb{R}$; $|ab| \leq \frac{1}{2}(a^2 + b^2)$, we have

$$
\begin{aligned}
|A(n)| &\leq \frac{n^{8HK-2}}{2} \sum_{i,j,k=0}^{n-1} \big| \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \ \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \big| \left(\sum_{l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \right) \\
&\quad + \frac{n^{8HK-2}}{2} \sum_{i,j,k=0}^{n-1} \big| \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \ \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \big| \left(\sum_{l=0}^{n-1} \langle \delta_{j/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \right) \\
&= n^{8HK-2} \sum_{i,j,k=0}^{n-1} \big| \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \ \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \big| \left(\sum_{l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 \right) \qquad (3.21)
\end{aligned}
$$

By (3.9) and (3.3), we obtain

$$
\begin{aligned}
n^{4HK} \sum_{l=0}^{n-1} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}}^2 &= \sum_{l=0}^{n-1} \theta^2(k,l) \\
&\leq 2^{1-2K} \left(\sum_{l=0}^{n-1} \gamma^2(k,l) + \sum_{l=0}^{n-1} \rho^2(k-l) \right) \\
&= 2^{1-2K} \left(\sum_{l=0}^{k} \gamma^2(k,l) + \sum_{l=k+1}^{n-1} \gamma^2(k,l) + \sum_{r=-k}^{n-1-k} \rho^2(r) \right) \\
&\leq 2^{1-2K} \left(\sum_{l=0}^{k} k^{4HK-4} + \sum_{l=1}^{n-1} l^{4HK-4} + 2 \sum_{r=0}^{n-1} \rho^2(r) \right) \\
&\leqslant 1 + \sum_{l=0}^{n-1} l^{4HK-4}. \qquad (3.22)
\end{aligned}
$$

On the other hand, by using (3.3)

$$
n^{4HK-2} \sum_{i,j,k=0}^{n-1} \big| \langle \delta_{i/n}, \delta_{j/n} \rangle_{\mathfrak{H}} \ \langle \delta_{i/n}, \delta_{k/n} \rangle_{\mathfrak{H}} \big| = \frac{1}{n^2} \sum_{i,j,k=0}^{n-1} |\theta(i,j)\theta(i,k)|
$$

$$
\begin{aligned}
&= \frac{1}{n^2}\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1}|\theta(i,j)|\right)^2 \\
&\leq \frac{1}{n^2}\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1}|\gamma(i,j)|+\sum_{j=0}^{n-1}|\rho(i-j)|\right)^2 \\
&= 2^{-2K}\frac{1}{n^2}\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}|\gamma(i,j)|+\sum_{j=i+1}^{n-1}|\gamma(i,j)|+\sum_{r=-i}^{n-1-i}|\rho(r)|\right)^2 \\
&\leq 2^{-2K}\frac{1}{n^2}\sum_{i=1}^{n-1}\left(i^{2HK-1}+\sum_{j=1}^{n-1}j^{2HK-2}+2\sum_{r=0}^{n-1}|\rho(r)|\right)^2 \\
&\trianglelefteq \frac{1}{n^2}\sum_{i=1}^{n-1}i^{4HK-2}+\frac{1}{n}\left(\sum_{j=1}^{n-1}j^{2HK-2}\right)^2. \qquad (3.23)
\end{aligned}
$$

By (3.21), (3.22) and (3.23),

$$
\begin{aligned}
|A(n)| &\trianglelefteq \frac{1}{n^2}\sum_{i=1}^{n-1}i^{4HK-2}+\frac{1}{n}\left(\sum_{j=1}^{n-1}j^{2HK-2}\right)^2 \\
&:= D(n). \qquad (3.24)
\end{aligned}
$$

If $0 < HK < \frac{1}{2}$,

$$
\begin{aligned}
D(n) &= \frac{1}{n^2}\sum_{i=1}^{n-1}i^{4HK-2}+\frac{1}{n}\left(\sum_{j=1}^{n-1}j^{2HK-2}\right)^2 \\
&\leq \frac{1}{n}\sum_{i=1}^{\infty}i^{4HK-3}+\frac{1}{n}\left(\sum_{j=1}^{\infty}j^{2HK-2}\right)^2 \\
&\trianglelefteq \frac{1}{n}. \qquad (3.25)
\end{aligned}
$$

If $\frac{1}{2} \leq HK < \frac{3}{4}$, then, by using the fact that for all $\alpha > -1$; $\sum_{k=1}^{n-1} r^{\alpha} \sim n^{\alpha+1}/(\alpha+1)$ as $n \to \infty$,

$$
\begin{aligned}
D(n) &= \frac{1}{n^2}\sum_{i=1}^{n-1}i^{4HK-2}+\frac{1}{n}\left(\sum_{j=1}^{n-1}j^{2HK-2}\right)^2 \\
&\leq \sum_{i=1}^{n-1}i^{4HK-4}+\left(\sum_{j=1}^{n-1}j^{2HK-\frac{5}{2}}\right)^2 \\
&\trianglelefteq n^{4HK-3}. \qquad (3.26)
\end{aligned}
$$

Combining (2.7), (3.20), (3.11), (3.25) and (3.26), we deduce that for every $0 < HK < \frac{3}{4}$,

$$d_{Kol}\left(V_n, N\right) \leqslant \begin{cases} \frac{1}{\sqrt{n}} & \text{if } HK \in \left(0, \frac{1}{2}\right] \\ \\ n^{2HK-\frac{3}{2}} & \text{if } HK \in \left[\frac{1}{2}, \frac{3}{4}\right) \end{cases}$$

Assume now that $HK = \frac{3}{4}$. From (3.21), (3.22) and (3.23) together with the fact that $\sum_{r=1}^{n-1} r^{-1} \sim \log(n)$ as $n \to \infty$,

$$\begin{aligned} \frac{|A(n)|}{\log^2(n)} &\leqslant \frac{1}{\log(n)}\left(\frac{1}{n^2}\sum_{i=1}^{n-1} i^{-1} + \frac{1}{n}\left(\sum_{j=1}^{n-1} j^{-\frac{1}{2}}\right)^2\right) \\ &\leqslant \frac{1}{\log(n)}, \end{aligned} \tag{3.27}$$

and this completes the proof of Theorem 3.1. □

4 Almost sure central limit Theorem

We are going now to prove the second main result of this paper, which state the ASCLT of the bifractional Brownian motion and its quadratic variation.

Proposition 4.1 *For all $H \in (0,1)$ and $K \in (0,1]$, we have, almost surely, for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,*

$$\frac{1}{\log(n)}\sum_{k=1}^{n}\frac{1}{k}\varphi(k^{-HK}B_k) \to \mathbb{E}\varphi(N), \quad \textit{as } n \to \infty,$$

where $N \sim \mathcal{N}(0,1)$.

Proof. The proof is straightforward by applying [2, Thorem 4.1 and Corollary 3.7] and the fact that

$$\begin{aligned} |E[B_jB_l]| &= 2^{-K}\left((j^{2H} + l^{2H})^K - |j-l|^{2HK}\right) \\ &\leq 2^{-K}\left(j^{2HK} + l^{2HK} - |j-l|^{2HK}\right) \\ &= 2^{1-K}|E[B_j^{HK}B_l^{HK}]|, \end{aligned}$$

where B^{HK} is a fractional Brownian motion with Hurst parameter HK. □

Theorem 4.1 *If $HK \in \left(0, \frac{3}{4}\right]$, then the sequence $(V_n)_{n\geq 0}$ satisfies the ASCLT. In other words, almost surely, for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$,*

$$\frac{1}{\log(n)}\sum_{k=1}^{n}\frac{1}{k}\varphi(V_k) \to \mathbb{E}\varphi(N), \quad \textit{as } n \to \infty,$$

where $N \sim \mathcal{N}(0,1)$.

Proof. We shall make use of Theorem 2.2. From Theorem 3.1, $(V_n)_n$ satisfies the CLT, so that, it remain to check conditions 1) and 2). The cases $HK \in \left(0, \frac{3}{4}\right)$ and $H = \frac{3}{4}$ are treated separately. By (3.6), we can write $V_n = I_2(g_n)$ where

$$g_n = \frac{n^{2HK}}{\sqrt{Var(Z_n)}} \sum_{k=1}^{n} \delta_{k/n}^{\otimes 2},$$

which implies that

$$g_n \otimes_1 g_n = \frac{n^{4HK}}{Var(Z_n)} \sum_{k,l=1}^{n} \langle \delta_{k/n}, \delta_{l/n} \rangle_{\mathfrak{H}} \; \delta_{k/n} \otimes \delta_{l/n}.$$

We deduce that

$$\|g_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}} = \frac{n^2}{Var^2(Z_n)} A(n). \tag{4.1}$$

Assume that $HK \in \left(0, \frac{3}{4}\right)$. Combining (3.11), (3.24), (3.25) and (3.26), we have

$$\|g_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}} \trianglelefteq (n^{-1} + n^{4HK-3}) \trianglelefteq \begin{cases} n^{-1} & \text{if } HK \in \left(0, \frac{1}{2}\right) \\ n^{4HK-3} & \text{if } HK \in \left[\frac{1}{2}, \frac{3}{4}\right) \end{cases}$$

Consequently, condition 1) in Theorem 2.2 is satisfied.
On the other hand, by (3.11), we have for $k < l$

$$\begin{aligned} \langle g_k, g_l \rangle_{\mathfrak{H}^{\otimes 2}} &= \frac{(kl)^{2HK}}{\sqrt{Var(Z_k)}\sqrt{Var(Z_l)}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \langle \delta_{i/k}, \delta_{j/l} \rangle^2_{\mathfrak{H}} \\ &\leq c_{H,K} \frac{1}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \theta^2(i,j) \\ &\leq c_{H,K} \frac{1}{\sqrt{kl}} \left[\sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho^2(i-j) + \left(\sum_{0 \leq i \leq j \leq k-1} + \sum_{i=0}^{k-1} \sum_{j=k}^{l-1} \right) \gamma^2(i,j) \right] \end{aligned}$$

As in the proof of [2, Theorem 5.1, page 1621], we obtain that

$$\frac{1}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho^2(i-j) \leq c_{H,K} \sqrt{\frac{k}{l}}.$$

Using Lemma 3.1, we obtain

$$\frac{1}{\sqrt{kl}} \sum_{0 \leq i \leq j \leq k-1} \gamma^2(i,j) \;\; \leq \;\; c_{H,K} \sqrt{\frac{k}{l}} \sum_{0 \leq i \leq k-1} i^{4HK-4} \leq c_{H,K} \sqrt{\frac{k}{l}}.$$

Again from Lemma 3.1, we have

$$\frac{1}{\sqrt{kl}}\sum_{i=0}^{k-1}\sum_{j=k}^{l}\gamma^2(i,j) \quad \leq \quad \frac{1}{\sqrt{kl}}\sum_{i=0}^{k-1}\sum_{j=1}^{l} j^{4HK-4} \leq c_{H,K}\sqrt{\frac{k}{l}}.$$

Combining all the above bounds we obtain

$$\langle f_k, f_l\rangle_{\mathfrak{H}^{\otimes 2}} \leq c_{H,K}\sqrt{\frac{k}{l}}.$$

Finally, condition 2) in Theorem 2.2 is satisfied.

Now, suppose that $HK = \frac{3}{4}$. It follows from (4.1), (3.12) and (3.27)

$$\|g_k \otimes_1 g_k\|^2_{\mathfrak{H}^{\otimes 2}} = \frac{k^2\log^2 k}{Var^2(Z_k)}\frac{A(k)}{\log^2(k)} \leq c_{H,K}\log^{-1} k.$$

Leads to

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{1}{n\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\|g_k\otimes g_k\|_{\mathfrak{H}^{\otimes 2}} \quad &\leq \quad c_{H,K}\sum_{n=2}^{\infty}\frac{1}{n\log^2 n}\sum_{k=1}^{n}\frac{1}{k\sqrt{\log k}}\\
&\leq \quad c_{H,K}\sum_{n=2}^{\infty}\frac{1}{n\log^{\frac{3}{2}} n} < \infty.
\end{aligned}$$

To close the proof, it suffices to show that

$$\langle g_k, g_l\rangle_{\mathfrak{H}^{\otimes 2}} \leq c_{H,K}\sqrt{\frac{k\log l}{l\log k}}, \qquad \forall k > l. \tag{4.2}$$

According to (3.12), we have

$$\begin{aligned}
\langle g_k, g_l\rangle_{\mathfrak{H}^{\otimes 2}} \quad &= \quad \frac{(kl)^{2HK}}{\sqrt{Var(Z_k)}\sqrt{Var(Z_l)}}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\langle\delta_{i/k},\delta_{j/l}\rangle^2_{\mathfrak{H}}\\
&\leq \quad \frac{c_{H,K}}{\sqrt{l\log k}\sqrt{k\log l}}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\theta^2(i,j)\\
&\leq \quad \frac{c_{H,K}}{\sqrt{l\log k}\sqrt{k\log l}}\left[\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\rho^2(i-j) + \left(\sum_{0\leq i\leq j\leq k-1} + \sum_{i=0}^{k-1}\sum_{j=k}^{l-1}\right)\gamma^2(i,j)\right]
\end{aligned}$$

As in the proof of [2, Proposition 6.4, page 1625], we have for all $1 \leq k \leq l$

$$\frac{1}{\sqrt{l\log k}\sqrt{k\log l}}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}\rho^2(i-j) \leq c_{H,K}\sqrt{\frac{k\log l}{l\log k}}.$$

Using Lemma 3.1 and the fact that $\sum_{r=1}^{n-1} r^{-1} \sim \log n$ as $n \to \infty$, we deduce that

$$\frac{1}{\sqrt{l \log k}\sqrt{k \log l}} \sum_{0 \leq i \leq j \leq k-1} \gamma^2(i,j) \quad \leq \quad c_{H,K} \frac{k \log l}{\sqrt{l \log k}\sqrt{k \log l}} \leq c_{H,K} \sqrt{\frac{k \log l}{l \log k}}. \tag{4.3}$$

Again from Lemma 3.1, we obtain

$$\frac{1}{\sqrt{l \log k}\sqrt{k \log l}} \sum_{i=0}^{k-1} \sum_{j=k}^{l-1} \gamma^2(i,j) \quad \leq \quad c_{H,K} \sqrt{\frac{k \log l}{l \log k}} \tag{4.4}$$

which complete the proof of the Theorem 4.1. □

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