Influence Of Mathematical Models On Warrant Pricing With Fractional Brownian Motion As Numerical Method

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ABSTRACT

Warrant pricing has become very crucial in the present market scenario. Different statistical and mathematical analysis of warrant evaluation to analyze the relationship between the prices of a warrants and common stocks and to take the price effects and probabilities into consideration to decide the fair value of warrants are used in this paper. Fractional Brownian motion is used to avoid independence on warrant pricing. Mathematical models are also applied on warrant pricing by using the Black-Scholes framework. The relationship between the price of the warrants and the price of the call accounts for the dilution effect is also studied mathematically in this paper.

Keywords: Warrant Pricing, fractional Brownian motion, Black-Scholes model, Option-pricing model, Dilution effects, Volatility, Mathematical analysis.

1 Introduction

Warrant is a kind of stock option which gives the holder the right but not the obligation to buy (if it is a call warrant) or to sell (if it is a put warrant) the stock or underlying asset by a certain date (for a European style warrant) or up until the expiry date (for an American style warrant) at a specified price (or strike price).

Warrants are classified as special options and can be divided into covered warrants and equity warrants according to the way they are issued. Covered warrants operate like options, only with a longer time frame and they are of American type. Covered warrants are typically issued by the traders and financial sectors and are for those who do not raise the company's stock after the day of expiration. Equity warrants are different from covered warrants because only the listed companies are recommended to issue them and the underlying assets are the issued stock of their company.

Black and Scholes (1973) state that their model can be used in many cases as an approximation to estimate the warrant pricing value and they used warrant pricing commonly as it was an extension of their call option model.

There are many complications in warrant pricing model. Black and Scholes (1973) mentioned that not only warrant pricing models have complications but also there are limitations inherent in the option pricing models. They investigated the error occurring when warrants are mistakenly priced as standard options ignoring the dilution effects. Therefore, it was very crucial to modify Black-Scholes call option model, because warrants are not written by other traders, they are provided by the company. Merton (1973) showed and proved that the Black-Scholes model can be modified to incorporate stochastic interest rates.

The volatility of warrant is described by the warrant pricing models, but under the framework of the existing pricing warrants analysis, the model based on stochastic volatility does not have an analytical solution. To this end, numerical method such as Fractional Brownian motion can be used to calculate the warrant pricing.

2 History of warrants

The long history of warrant pricing began very early. Warrant pricing was not usually the financial theory property. Lot of researchers were focusing on the option pricing because warrant pricing was complicated than option pricing.

Mandelbrot and Taylor (1967) observed that there are fractal behaviour in stock prices. Sidney (1949) released a warrant survey book "The Speculative Merits of Common Stock Warrants". It was regarded as the first book to reveal the common stock warrants which turn in the most spectacular performance of any group of securities and this common stock warrants are very huge.

McKean (1965) and Samuelson (1965) showed the warrant valuation which consider the non-negative value to the warrants holder who has the right to exercise a warrants at any time (being an American warrant) before its maturity.

A crucial influence in 1970s research on warrant pricing was the work of Chen (1970). He gave the equation of warrants expected value on its exercising date and derive an equation to value warrants by making use of dynamic programming technique. His work aligned with Sidney's work by comparing the perpetual warrants (warrants with indefinite length of life) with common stocks. Chen affirmed that the perpetual warrants cannot be worth more than the common stocks because the company which owns the perpetual warrants are exercisable at zero exercise price which is the same as owning common stocks.

The market price of the stock is always below the exercise price at the time of issue. The most popular method for valuing options are based on the Black and Scholes (1973) and Merton (1973) models. Their models for pricing options have been taken into consideration to many different commodities and payoff structures and they have become the most popular method for valuing options and warrants.

Black and Scholes derived their formulas and assumed that the option price is the function of the stock price. It is noted that the changes in the option price are completely correlated with the changes in the stock price. Black and Scholes (1973) showed how their formulas can be modified to value European warrants. Merton's model is the same as Black-Sholes model despite that the maturity for default free bond which matures at the same time as the options' expiration date is used for the interest rate.

Merton's model of the option pricing was not appropriate for warrant pricing because he assumed that the variance of the default free bond is constant and the variance of bond prices may change due to long life of warrants. European call option was the easiest one in the stock options.

Galai and Schneller (1978) derived the value of the warrants and the value of the company that issues warrants by discussing the equality between the value of the warrants and the value of the call options on a share of the company which warrants hold for any other financial or investment decisions of the company. Several studies on warrants have ignored the dilution effects and equated the warrants to the call options.

Lot of researchers measured warrant's life comparing with option's life and found that warrants have a long life. Kremer and Roenfeldt (1993) used jump-diffusion more often to price warrants. There is a high possibility that the stock price might jump during the life of warrants. These diffusions of the stock returns are more relevant for warrant pricing than for option pricing. Jump diffusion is listed as a bias model for pricing options, but it is more efficient for pricing warrants.

3 Warrant pricing vs. Option pricing

Warrant pricing and option pricing carry the right to buy the shares of an underlying asset at a certain price and can be exercised anytime during their life (if they are of American style) or on expiration date (if they are of European style). While the call options are issued by an individual, the warrants are issued by a company. Warrants proceeds increase the company's equity and when it is time to exercise them, new shares are always issued and the payment of cash increases the assets of the issuing company because of the dilution of equity and dividend. When options are exercised, the shares can come from another investor or public exchange.

In warrant pricing, many researchers ignored the dilution effects and valued warrants as the call options on common stocks of the company. The valuation of warrants and call options involves making assumptions about the capital structure of the company and future dividend policy.

The call options can uniquely be priced and the price can be independent in the amount of written call options, given the fact that all call options can be exercised simultaneously, each call option is a separate stake. Nevertheless, when warrants are exceptional, they can be exercised and new shares can be formed and the changes in the capital structure of the company and dividend policy can occur. Warrant and option pricing are based on the underlying asset such as stocks and bonds. Researchers have used the following formulas for pricing options and warrants and to study the dilution effects.

3.1 Formula for pricing options

The Black and Scholes (1973) option pricing model specifies the following price for a call and put option on a nondividend-paying stock

$$C = SN(d_1) - Xe^{-r(T-t)}N(d_2), \tag{3.1}$$

$$P = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{3.2}$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

C is the value of the call option, P is the value of the put option, S is the price of the underlying stock, X is the exercise price of the call and put option, r is the annualized risk-free interest rate, $T-t$ is the time until expiration, σ is the annualized standard deviation of the logarithmic stock return, and $N(\cdot)$ is the probability from the cumulative standard normal distribution.

3.2 Formula for pricing warrants

Galai and Schneller (1978) presented the first solution for the warrant pricing problem in which they incorporated the dilution effect by deriving the following equation:

$$W = \frac{N}{N+n}C_w, \tag{3.3}$$

where
W is the value of the warrant, N is the number of shares outstanding, n is the number of new shares to be issued if warrants are exercised, C_w is the value of a call option written on the stock of a firm without warrants.
The equation (3.3) is based on the assumption that the company with capital structure consists only equity warrants and it is defined as

$$V = NS + nW, \tag{3.4}$$

where
V is the value of the company's equity, S is the stock price.
It is assumed in Equation (3.4) that not only the value of the company's stock follows the diffusion process, but also the value of the company's equity (V).

Schulz and Trautmann (1994) have compared the outcomes of the original Black-Sholes model with the outcomes of the correct warrant valuation model and they concluded that although the high dilution effects are assumed, the Black-Sholes models produce small biases. Crouhy and Galai (1991) note that warrant prices are always calculated by multiplying the outcome from the option pricing model (such as the Black-Scholes model) by the dilution effects $\left(\frac{N}{N+n}\right)$.

If the standard deviation of the return in the company's equity is constant, it leads to the following equation [4]:

$$C_w = \widehat{N}[\widehat{S}N(d_1) - Xe^{r(T-t)}N(d_2)], \tag{3.5}$$

where
$d_1 = \frac{\left[\ln\left(\frac{\widehat{S}}{X}\right) + (T-t)(r+\frac{1}{2}\sigma^2)\right]}{\sigma\sqrt{T-t}}$,
$d_2 = d_1 - \sigma\sqrt{T-t}$,
$\widehat{N} = \frac{N}{\left[\left(\frac{N}{k}\right)+M\right]}$,
$\widehat{S} = S + (\frac{M}{N})W - PV_D$,

W is the value warrant price, N is the number of outstanding shares, k is the number of shares that can be purchased with each warrant, M is the number of outstanding warrants, S is the stock price, PV_D is the present value of dividends expected over the life of the warrant, X is the warrant exercise price, r is the risk-free interest rate, σ is the firm-value process volatility, T is the time to expiration on the option, and $N(d)$ is the probability that a standard normal variable will take on a value less than equal to d.

4 Mathematics models on warrant pricing

Zhang et al. (2009) priced equity warrants using fractional Brownian motion. They denoted company's equity by V_T at time T, saying that the company will receive a cash inflow from the payment of the exercise price of MlX. If warrant holders exercise the warrants, the value of the company's equity will increase to $V_T + MlX$. This value is distributed among $N + Ml$ shares so that the price of share after exercise becomes

$$\frac{V_T + MlX}{N + Ml},$$

where
N is the number of shares of outstanding stocks, M is the number of warrants issued, l is the number of shares of stock that can be bought with each warrant, and X is the strike price of option.
The warrants can be exercised only if the payoff is greater than minimum guarantee provision, i.e.,

$$l\left(\frac{V_T + MlX}{N + Ml} - X\right) > B, \tag{4.6}$$

where B is the minimum guarantee provision. This shows that the warrants value at expiration time satisfies

$$\begin{aligned} W_T &= l \max\left[\frac{V_T + MlX}{N + Ml} - \left(X + \frac{B}{l}\right), 0\right] + B, \\ &= \frac{Nl}{N + Ml} \max\left[\frac{V_T}{N} - X - \frac{N + Ml}{Nl} B, 0\right] + B. \end{aligned}$$

Letting $\alpha = \frac{M}{N}$ and $\hat{X} = X + \frac{N+Ml}{Nl}B$, above implies

$$W_T = \frac{l}{1 + \alpha l} \max\left(\frac{V_T}{N} - \hat{X}, 0\right) + B. \tag{4.7}$$

Since V_T denotes the company's equity (including the warrants) at time T. We have, $V_T = NS_T + MW_T = NS_T + \alpha NW_T$. Setting $\hat{S_T} = S_T + \alpha W_T$, equation (4.7) implies

$$W_T = \frac{l}{1 + \alpha l} \max(\hat{S_T} - \hat{X}, 0) + B. \tag{4.8}$$

In the fractional Brownian motion and risk-neutral world, the price at every $t (t \in [0, T])$ of an equity warrant with strike price X and maturity T is given by

$$W_t = \frac{l}{1 + \alpha l} [\hat{S}_t N(d_1) - \hat{X} e^{-r(T-t)} N(d_2)] + B e^{-r(T-t)}, \tag{4.9}$$

where

$$d_1 = \frac{\ln \frac{S_t}{X} + r(T - t) + \frac{\sigma_V^2}{2}(T^{2H} - t^{2H})}{\sigma_V \sqrt{T^{2H} - t^{2H}}},$$

and

$$d_2 = \frac{\ln \frac{S_t}{X} + r(T - t) - \frac{\sigma_V^2}{2}(T^{2H} - t^{2H})}{\sigma_V \sqrt{T^{2H} - t^{2H}}},$$

where
r is the risk-free interest rate, $T-t$ is the time to expiration of warrant, σ_V is the firm-value process volatility, H is the Hurst parameter, α denotes the percentage of warrants issued in shares of stock outstanding, and $N(\cdot)$ is the cumulative probability distribution function of a standard normal distribution.

5 Warrant pricing using fractional Brownian motion

Many authors used fractional Brownian motion to avoid independence on warrant pricing. The name fractional Brownian motion is firstly seen in the work of Mandelbrot and Van Ness (1968). Below is the formal definition of fractional Brownian motion and its properties.

Definition 5.1 (Biagini et al. (2008)) *Let $H \in (0,1)$ be a constant. A fractional Brownian motion $(B^{(H)}(t))_{t\geq 0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function*

$$E[B^{(H)}(t)B^{(H)}(s)] = \tfrac{1}{2}t^{2H} + s^{2H} - |t-s|^{2H}, \forall s,t \in R^{+}.$$

For $H = \frac{1}{2}$, the fractional Brownian motion is a standard Brownian motion. By definition, a standard fractional Brownian motion $B^{(H)}$ has the following properties:

- $B^{(H)}(0) = E[B^{(H)}(t)] = 0$, for all $t \geq 0$;
- $B^{(H)}$ has homogeneous increments, i.e., $B^{(H)}(t+s) - B^{(H)}(s)$ has the same law of $B^{(H)}(t)$ for $s, t \geq 0$;
- $B^{(H)}$ is a Gaussian process and $E[B^{(H)}(t)^2] = t^{2H}, t \geq 0$, for all $H \in (0,1)$;
- $B^{(H)}$ has continuous trajectories .

Parameter H of $B^{(H)}$ was named by Mandelbrot after the name of the hydrologist Hurst, who made a statistical study of yearly water run-offs of the Nile river. Mandelbrot (1983) used this process to model some economic time series. Most recently these processes have been used to model telecommunication traffic [7].
The values of the Hurst exponent range from zero to one. In [15] it is mentioned that

- $H = \frac{1}{2}$ or close to that value indicate a random walk or a Brownian motion. In this case no correlation is present between any past, current, and future elements. In other words, there is no independence behaviour in the series. Such series is not easy to predict.
- $H < \frac{1}{2}$ indicates the presence of anti-persistence, meaning that if there is an increase, the decrease will automatically follow and vice versa. This behaviour is also called the mean reversion in the sense that the future values will always tend to return to a longer term mean value.
- $H > \frac{1}{2}$ indicates the presence of the persistence behaviour, meaning that the time series is trending. It may be a decreasing or increasing trend.

Fractional Brownian motion has two crucial properties: self-similarity and long-range dependence. The self-similarity is $a > 0$ then $(B^{(H)}(at), t \geq 0)$ if $(a^{(H)}B^{(H)}, t \geq 0)$. The long range means that if $r(n) = E[B^{(H)}(t)(B^{(H)}(n+1) - B^{(H)}(n))]$ then $\sum_{n=1}^{\infty} r(n) = \infty$. These two properties make the fractional Brownian motion a suitable instrument in different applications such as mathematical finance.
The assumptions which are used to derive the warrant pricing formula in fractional Brownian motion are as follows (see [16] for further details):

(i) The warrant price is the function of the time and underlying stock's price,

(ii) The shorting of assets with all use of proceeds is allowed,

(iii) There are no transactions costs or taxes and all securities are perfectly divisible,

(iv) Risk less arbitrage opportunities are controlled,

(v) The trading of the asset is continuous,

(vi) The risk-free rate of interest and all the maturities is constant,

(vii) The price of the stock follows fractional Brownian motion process and the dynamics of the risk adjusted process $(S_t, t \geq 0)$ are defined as

$$dS_t = S_t(\mu dt) + \sigma_v dB^{(H)}(t), 0 \le t \le T, \tag{5.10}$$

where
$B^{(H)} = B^{(H)}(t, x), t > 0$ is the Fractional Brownian motion, μ is the expectation of the yield rate, σ_v is the firm-value process volatility, T is the option expiration time, S_t is the stock price at time t.
Hu and ⊘ksendal (2000) made use of Itô integrals with respect to $B^{(H)}$ and showed that the fractional Black-Scholes market presents no arbitrage opportunity.

6 Numerical results

In this section some results that are calculated for warrant pricing are illustrated.
Zhang et al. (2009) used the data of Changdian warrants collected from 25 May 2006 to 29 January 2007 (the expiration date) and considered the probability distribution. The yield series distribution of Changdian warrants are greater than zero, which implies that the yield series distribution of Changdian warrants are not normally distributed.

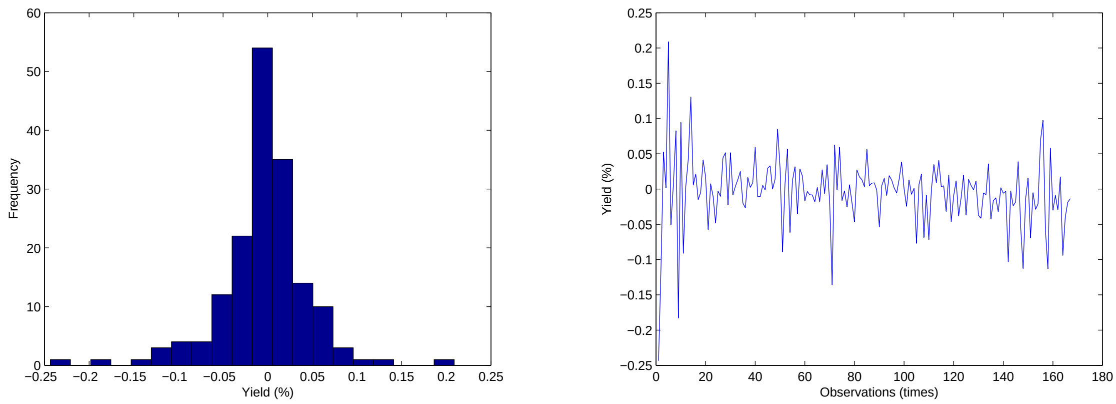

Figure 1: Histogram and Plot of Changdian warrant pricing returns from 25 May 2006 to 29 January 2007 and its probability distribution. [Programmed in MATLAB and reproduced by using the data from Zhang et al. (2009)]

Histogram and plot show the Changdian warrants which are the first warrants in China, while histogram shows a bell-shaped and symmetrical histogram with data points equally distributed around the middle. The graph is skewed to the right and kurtosis is greater than three which implies that the yield series of Changdian warrants is leptokurtic. A plot shows the independent variable of observation times, which has high volatility to the percentage of yield no matter what happens. It is hard to see the consistent pattern in this figure. This gives the insight of using fractional Brownian motion. Zhang et al. (2009) developed the fractional Browian motion considering the mathematical models of strong correlated stochastic processes.

Using the same data, the descriptive statistics of stock price and warrant price using the same Changdian market is compared below. Table 1 shows the Changdian stock price which was left out when simulating warrant pricing [16]. The logarithmic returns is defined as

$$x_t = \ln \frac{y_{t+1}}{y_t} = \ln y_{t+1} - \ln y_t,$$

where y_t is the closing quotation of Changdian stocks and warrants at time t.
The yield distribution of Changdian stocks is greater than zero, which implies that the yield distribution of Changdian stocks is not normal distribution like in Changdian warrants. It performs skew distribution and its Kurtosis is greater than three obviously, which implies that the yield of Changdian stocks is also leptokurtic. The value of Jarque-Bera in Changdian stocks implies that the yield distribution is less probability group near the starting point and in the tails. While the value of Jarque-Bera in changdian warrants implies that the

Table 1: The yield series of Changdian stocks' descriptive statistics

Observations	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera
168	0.0030	0.0197	0.4393	2.2126	9.7436

Table 2: The yield series of Changdian warrants' descriptive statistics

Observations	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera
168	0.0042	0.0490	0.5773	8.6698	234.3679

yield distribution of Changdian warrants have more probability group near the starting point and in the tails.

Table 3: Results of regression analysis and statistical testing for the Changdian warrants and stocks

Observations	Multiple R	R^2	Adjusted R^2	Standard error
168	0.8762	0.7677	0.7663	0.4335

It is crucial to expand the descriptive statistics of Changdian warrants and Changdian stocks and calculate regression analysis and Anova. Letting Changdian warrants to be dependent variables (Y) and Changdian stocks to be independent variables (X) we compute Table 3. It shows the multiple regression of 0.8762 which is a strong correlation coefficient. R^2 is equals to 0.7677 which means that 77% of the variance is shared between Changdian warrants and stocks.

7 Conclutions

Different methods are shown for pricing warrants. The warrant pricing models are based on the variables used in the Black-Scholes option pricing formula. The warrant pricing has been compared with option pricing theoretically and practically, showing the similarities and how they differ. The results have shown that the Black-Scholes model associated for dilution as the stock price (S) are replaced by the value of the company (V). The standard deviation of the stock's return (σ) is replaced by the standard deviation of the value (σ_v) and the outcome model is multiplied by the dilution factor $(1/(1+q))$.

In order to generate the initial approximation to the warrant pricing, a certain numerical method was also used, like *fractional Brownian motion*, which is used by many authors to avoid independency. To derive warrant pricing formulas in fractional Brownian motion the assumptions and fractional Black-Scholes formula were taken into consideration. The warrant pricing in fractional Brownian motion is similar to the European call option.

It is shown that if warrant prices are calculated twice using the Black-Scholes model, they give the same results, but if using fractional Brownian motion they give different results because of the long memory property.

In fractional Brownian motion, the Hurst exponent is utilised, which is a tool used to test the memory in time series, and therefore helps to determine the behaviour and efficiency of the markets. A Hurst exponent which is equal to $\frac{1}{2}$ indicates the independence behaviour of the series, whereas the Hurst exponent values different from $\frac{1}{2}$ show the presence of long memory or long range dependence which is characterised by the fractional Brownian motion model.

Some of the methods discussed above are purely for options but after necessary modifications, they can be used to price warrants. Such a modification is being done but due to time limitations, I would explore them further in the near future.

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