Maximally Robust Designs for Two-Level Main-Effect Plans

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Abstract

In this study, a new criterion called the minimum breakdown criterion is employed to assess the robustness of two-level main-effect plans against missing data. Based on this criterion, a special class of maximally robust robust designs, called the minimum breakdown designs, is proposed as the planning schemes, when the observations tend to be missing. Most interestingly, a minimum breakdown design provides the highest probability to obtain a nonsingular residual design, when a subset of observations is unavailable. A series of minimum breakdown designs is generated by an exhaustive computer search, and a catalogue of the proposed designs with practical run-sizes is presented for real-life experiments.

Key words: Hadamard matrix; Missing value; Orthogonal array; Robust design.

1 Introduction

Two-level full factorial and fractional factorial designs have been successfully used in many scientific disciplines, such as industrial process/product improvements, social sciences, biomedical and clinical trials, etc. Typically, the design resolution and minimum aberration criteria are used for selecting designs against the aliasing among the lower oder factorial effects. The interested reader is referred to Wu and Hamada (2009) for a comprehensive introduction to the theory and applications of fractional factorial designs. Alternatively, if the experimental responses of certain treatment combinations tend to be missing, the incomplete data usually lead to a loss in estimation efficiency, or the most critical situation that the factorial effects of interest are no longer estimable. Therefore, the design robustness against missing data should be taken into account in the planning stage of a multifactorial experiment, when the observations tend to be unavailable.

Planning a robust design against missing data was first investigated by Ghosh (1979). Dey (1993) termed the concept proposed by Ghosh (1979) as the Criterion-1 robustness. A two-level design is said to be Criterion-1 robust against the loss of i observations, if all of the potential residual designs are nonsingular. A design is said to be nonsingular for $\boldsymbol{\beta}$, a collection of factorial effects of interest, if the corresponding information matrix is nonsingular. The reader can consult John (1979), Ghosh (1980), Moore (1988), MacEachern *et al.* (1995) for some results regarding the design robustness on two-level factorial designs. However, when ranking designs with respect to the Criterion-1 robustness, there could be several candidate designs robust against the loss of i observations. Consequently, a new criterion is required for further discriminating those designs of equal performance. Recently, Tsai and Liao (2013) proposed a new criterion called the minimum breakdown criterion for addressing this issue. Based on the minimum breakdown criterion, they proposed a series of robust designs in blocks of size two for single-factor experiments. In this study, the main focus is on the assessment of two-level main-effect plans by the minimum breakdown criterion.

2 Minimum Breakdown Criterion

First, the minimum breakdown criterion, which can be regarded as a natural generalization of the Criterion-1 robustness, is introduced as follows.

Definition 1. *Let d be an N-run main-effect plan for n factors, and M_i represent the number of residual designs, which are singular for μ and all the main effects, among the $\binom{N}{i}$ possibilities, when any set of i observations is deleted from d. Then the singular pattern of d is defined as $\boldsymbol{M}(d) = (M_1, M_2, \ldots, M_{N-n-1})$.*

Apparently, the singular pattern provides more information than the Criterion-1 robustness. Criterion-1 robustness only indicates the maximal number of removable observations that a residual design remains nonsingular. The singular pattern provides the numbers of singular designs for various numbers of missing observations. Note that a design d is Criterion-1 robust against the loss of i observations, if $M_1 = M_2 = \cdots = M_i = 0$. Under the assumption that each treatment combination has an equal chance to be missing, the probability to derive a nonsingular residual designs is given by

$$P_i = 1 - \frac{M_i}{\binom{N}{i}},$$

when i runs are removed from d. Clearly, maximizing P_i is equivalent to minimizing M_i. In the planning stage of a multifactorial experiment, it is natural to choose a design with P_i's as large as possible, that is, with M_i's as small as possible. Based on the singular pattern, a class of maximally robust designs against missing data is now defined as follows.

Definition 2. *For any two designs d_1 and d_2, let q be the smallest integer such that $M_q(d_1) \neq M_q(d_2)$. A design d_1 is said to be less breakdown than d_2, if $M_q(d_1) < M_q(d_2)$. If there is no design with less breakdown than d_1, then d_1 is called the minimum breakdown design.*

According to Definition 2, a minimum breakdown design is the one which has the greatest q, such that the M_i's are sequentially minimized for $1 \leq i \leq q$. In other words, the corresponding P_i's are sequentially maximized. Namely, it provides the highest probability to derive a nonsingular residual design, when i observations are missing.

3 Results and Discussions

Typically, a two-level orthogonal main-effect plan for n factors can be readily obtained by projecting n factors onto $\boldsymbol{H}_N$, a semi-normalized Hadamard matrix of order N. Note that the first column of $\boldsymbol{H}_N$, which is a vector of all entries equal to 1, is excluded from the projection here. Alternatively, two Hadamard matrices are said to be non-isomorphic, if one cannot be obtained from the other through permutations of rows or columns, or switching the symbols for each column, or a combination of the above. Based on the catalogue of the non-isomorphic Hadamard matrices presented on the website of N. J. A. Sloane (http://neilsloane.com/hadamard/index.html), an exhaustive computer search is implemented to generate the minimum breakdown designs over two-level orthogonal main-effect plans for $N/2 \leq n \leq N-2$ where $N = 12$, 16 and 20. For the ease of presentation, only a part of the obtained designs are displayed here in Tables 1, the other designs are available upon request from the authors.

When the run-size is significantly larger than the number of parameters of interest, the missing data usually do not have a severe impact on the estimation efficiency. Therefore, the exhaustive computer search is implemented to generate only the designs for $N/2 \leq n \leq N-2$ where $N = 12$, 16 and 20. In addition, it becomes computationally infeasible for the cases of $N \geq 24$, due mainly to a large number of non-isomorphic Hadamard matrices of the same order. For example, there are 60 and 487 non-isomorphic Hadamard matrices for $N = 24$ and $N = 28$, respectively. For constructing the minimum breakdown designs for $n \geq 24$, an efficient algorithm is on requirement to carry out the search.

Table 1: Minimum breakdown designs for two-level main-effect plans.

n	N	$\boldsymbol{H}$	Columns selected
6	12	Had.12	$\{2,3,4,5,6,7\}$
7	12	Had.12	$\{2,3,4,5,6,7,8\}$
8	12	Had.12	$\{2,3,4,5,6,7,8,9\}$
8	16	Had.16.2	$\{2,3,4,5,6,7,9,16\}$
9	16	Had.16.1	$\{2,3,4,5,6,7,9,11,15\}$
10	16	Had.16.0	$\{2,3,4,5,6,7,9,10,11,13\}$
10	20	Had.20.2	$\{2,3,4,5,6,7,8,9,11,15\}$
11	20	Had.20.3	$\{2,3,4,5,6,7,9,11,13,14,17\}$
12	20	Had.20.1	$\{2,3,4,5,6,7,8,9,10,11,12,13\}$
13	20	Had.20.1	$\{2,3,4,5,6,7,8,9,10,11,12,13,15\}$
14	20	Had.20.1	$\{2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$
15	20	Had.20.1	$\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$

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