RIGHT CENSORED RAYLEIGH MODEL: SHRINKAGE AND RELIABILITY ESTIMATORS*

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The Rayleigh distribution (as a special case of the Weibull distribution), and essentially its censored counterpart, is considered such as to derive objective estimators for various quantities. [1], [2], and [4] derived and juxtaposed the Bayesian estimators under certain loss functions, with specific reference to the Jeffreys prior and other objective prior choices. This presentation aims to generalize the chosen objective prior distribution to derive estimators for the parameter under each loss function considered, all in the Bayesian paradigm. Shrinkage estimators and reliability estimators are also determined and their effect studied. The loss function as proposed by [3] is compared to the well-known squared error loss, and the general entropy loss is compared to the linear exponential loss (LINEX) (stemming from reasoning by [5]. These estimators are compared to each other subsequently via a Monte Carlo simulation approach. A short application is discussed and results are summarized.

1. Introduction

1.1. *Objective priors e.a.*

The Rayleigh distribution, and in particular the censored model, is widely known for its extensive applications in reliability theory and communications engineering. In [2] a family of quasi-density priors was introduced, namely of the form $g(\theta) = 1/\theta^m, m > 0$, for a model where the parameter θ is of interest. It can be shown that this family of priors include the Jeffreys prior amongst others. This θ is also the parameter of interest for the censored model. This prior density was used in obtaining the posterior distribution, and hence the estimators, under the censored Rayleigh model (as was done in [4]).

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As [1] showed, this model collapses into closed form expressions and is comfortable to work with stemming from the nature of the posterior distribution – which is shown to be gamma. Next, a discussion regarding the estimators follows.

1.2. *Loss functions & estimators*

The well-known squared error loss has been considered widely throughout most statistical literature and has been compared various times with the linear exponential (LINEX) loss. Here it is compared to the loss proposed by [3], whereas the LINEX loss is compared to the general entropy loss. The functions, together with their respective estimators, are given below in Table 1:

Table 1. Loss functions & corresponding estimators.

Loss function name	Loss function	Parameter estimator	Survival function
Squared error loss	$L(\theta,\hat{\theta}) = (\theta - \hat{\theta})^2$	$\hat{\theta} = \frac{d-m+1}{T}$	$\widehat{S(t)} = e^{-\hat{\theta}t^2}$
Al-Bayyati loss	$L(\theta,\hat{\theta}) = \theta^c(\theta - \hat{\theta})^2$	$\hat{\theta} = \frac{d-m+1+c}{T}$	
LINEX loss	$L(\theta,\hat{\theta}) = e^{a\Delta} - a\Delta - 1$	$\hat{\theta} = \frac{d-m+1}{a} ln\left[1+\frac{a}{T}\right]$	
GEL	$L(\theta,\hat{\theta}) \propto \left(\frac{\hat{\theta}}{\theta}\right)^p - pln\left(\frac{\hat{\theta}}{\theta}\right) - 1$	$\hat{\theta} = \sqrt[p]{\frac{\Gamma(d-m+1)}{\Gamma(d-m+1-p)T^p}}$	

with restrictions $a \neq 0, p \neq 0, t > 0, and\ c \in \mathcal{R}$. T is a statistic from the sample and *d* is the number of observed lifetimes.

These estimators have been compared to each other w.r.t changing the prior degree, namely *m*. In figure 1, the change in the expected loss can be observed by increasing this prior degree for the SEL and the Al-Bayyati loss, and in figure 2 similarly for LINEX loss and GEL.

Figure 1. Risk (expected loss) for SEL and Al-Bayyati loss, for different degrees of *m.*

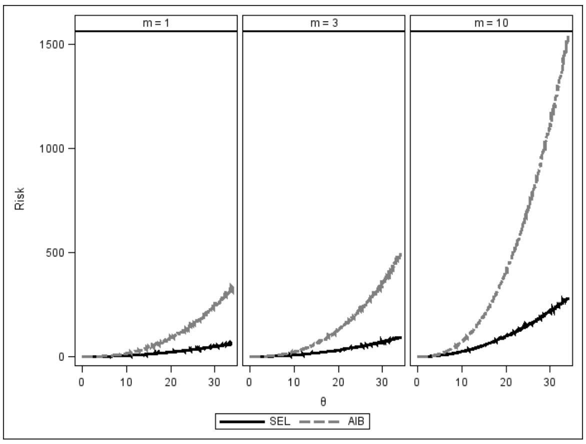
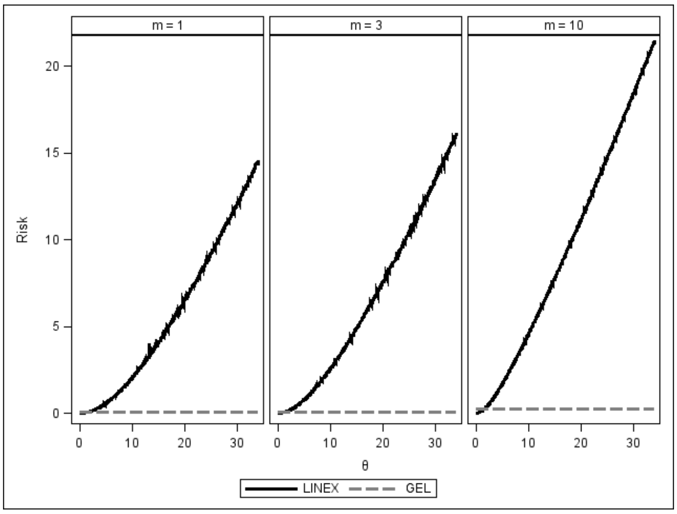


Figure 2. Risk (expected loss) for LINEX loss and GEL, for different degrees of *m*



1.3. *Risk efficiency*

The risk efficiency is also determined for both groups of loss functions. This method provides an intuitive way of determining which estimator – under a certain loss function – performs more admirable than the other. The risk needed to be considered is:

$$R_L^*(\hat{\theta}) = E_T\left(L(\hat{\theta}, \theta)\right)$$

collapsing then in a risk efficiency of form:

$$RE_L(\widehat{\theta_L}, \widehat{\theta_y}) \equiv \frac{R_L^*(\widehat{\theta_y})}{R_L^*(\widehat{\theta_L})}$$

where L denotes the loss function under consideration and its respective estimator, and y denotes the other estimator.

Closed form expressions are obtained for the risk efficiency between SEL and ABL under SEL, and also for ABL and SEL under ABL. These expressions are also shown to be independent of the Rayleigh parameter. For the risk efficiency between LINEX loss and GEL under LINEX loss, an approximation is obtained and illustrated.

1.4. *Shrinkage estimation*

Shrinkage estimation is a method that a naïve or target estimator is improved, in some sense, by combining it with other information, such as a prior point guess of the parameter(s) of interest. The shrinkage methodology has been applied in different situations, including estimation mortality rates and improved estimation in sample surveys and Weibull Type-II censoring studies. A shrinkage estimator for the parameter θ [6] when a prior point guess value θ_0 *of* θ is available, is defined as $S = k\hat{\theta} + (1-k)\theta_0, 0 \leq k \leq 1$. These similar expressions are derived for each loss function and their risk efficiencies determined, and consequently compared between each other.

1.5. *Conclusion*

Different objective estimators have been obtained for the censored Rayleigh model. Specific attention has been given to the expected loss of each loss function, the risk efficiency between certain functions and attempts at arriving at more suitable estimators have been made via shrinkage estimator investigations. Recommendations have been made and the results thoroughly discussed.

References

[1] P.J. Mostert. A Bayesian method to analyze different cancer lifetimes using Rayleigh models. *Unpublished PhD thesis.* UNISA. South Africa. (1999)

[2] A.A. Soliman. Comparison of LINEX and quadratic Bayes estimators for the Rayleigh distribution. *Communications in Statistics – Theory and Methods* 29:1, p. 95 – 107 (2000).

[3] H.N. Al-Bayyati. Comparing methods of estimating Weibull failure models using simulation. *Unpublished PhD thesis*. Baghdad University, Iraq (2002).

[4] J.T. Ferreira. Objective Bayesian estimators for the right censored Rayleigh model using different loss functions. *Unpublished Honours Essay*, University of Pretoria, South Africa (2012).

[5] R. Calabria & G. Pulcini. Point Estimation under asymmetric loss function as for left-truncated exponential samples. *Communications in Statistics – Theory and Methods* 25:3, p. 585-600 (1996).

[6] G. Prakash & D.C. Singh. A Bayesian shrinkage approach in Weibull Type-II censored data using prior point information. *REVSTAT*, 7:2, p. 171-197 (2009).