

Measures of Portfolio Credit Risk by Applying Asymptotic Single-Risk Factor Model

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Abstracts

The Asymptotic Single-Risk Factor (ASRF) model underpins the capital calculations of the internal ratings-based (IRB) approaches of the Basel II framework (Basel Committee on Banking Supervision 2005). It turned out that portfolio invariance of the ASRF model is a property with strong influence on its application on credit portfolio models. The ASRF models are derived from ordinary credit portfolio models by the law of large numbers and known to be portfolio invariant. When a portfolio consists of a large number of relatively small exposures, idiosyncratic risks associated with individual exposures tend to cancel out one-another and only systematic risks that affect many exposures have a material effect on portfolio losses. In the ASRF model, all systematic risks, which affect all borrowers to a certain degree, are modeled with only one systematic risk factor. In this paper, we describe the modeling framework of the ASRF model and carry out an implementation of the ASRF model by means of a standard programming tool, namely the Microsoft Visual Basic for Applications, for measuring the portfolio credit risk of a sample portfolio.

Keywords: probability of default, loss given default, credit Value-at-Risk, expected shortfall, internal ratings-based approach, multi-state model

1. Introduction

In the specification of the Basel II model, the property of portfolio invariance¹ has a strong influence on the structure of the portfolio model. Gordy (2003) showed that the Asymptotic Single-Risk Factor (ASRF) models of portfolio credit risk are portfolio invariant. The ASRF models are derived from ordinary credit portfolio models by the law of large numbers. When a portfolio consists of a large number of relatively small exposures, idiosyncratic risks associated with individual tend to cancel out and only systematic risks have a material effect on portfolio losses. In the ASRF model, all systematic risks are modeled with only one systematic risk factor. In this paper, we review the ASRF modeling framework. Then, we carry out an implementation of the ASRF model by Microsoft Excel's Visual Basic Application (VBA), for assessing portfolio credit risk of a sample bond fund portfolio.

2. Methods

In this section, we outline the framework of ASRF model and discuss some key assumptions for modeling portfolio credit risk. Then, we develop VBA program for quantifying and comparing portfolio credit risks generated by alternative models.

2.1. The ASRF model

The ASRF model of portfolio credit risk was introduced by Vasicek (1991). He

¹ The IRB risk-weight functions of Basel II were developed with the idea that they would be portfolio invariant, i.e. the capital required for any given loan should only depend on the risk of that loan and must not depend on the portfolio it is added to. This characteristic has been deemed vital in order to make the IRB applicable to a wider range of countries and institutions.

postulates that an obligor defaults when its assets fall below some threshold. In addition, the model assumes that asset values are driven by a single common factor

$$V_{iT} = \rho_i \cdot M_T + \sqrt{1 - \rho_i^2} \cdot Z_{iT} \tag{1}$$

where V_{iT} is the value of assets of obligor i at time T ; M_T and Z_{iT} denote the common and idiosyncratic factors respectively; and $\rho_i \in [-1, 1]$ is the obligor-specific loading on the common factor. The common and idiosyncratic factors are independent and scaled to random variables with mean 0 and variance 1.

The ASRF model delivers a closed-form approximation to the probability distribution of default losses on a portfolio of N exposures. The accuracy increases when the number of exposures grows, $N \rightarrow \infty$, and the largest exposure weight shrinks, $\sup_i(w_i) \rightarrow 0$. In these limits, in which the portfolio is perfectly granular, the probability distribution of default losses can be derived as follows.

2.1.1. Obtaining the conditional default probability of an obligor in the portfolio

Let the indicator I_{iT} equal 1 if obligor i is in default at time T and 0 otherwise. Conditional on the value of the common factor, the expectation of this indicator equals

$$E(I_{iT}|M_T) = \Pr(V_{iT} < F^{-1}(PD_{iT})|M_T) = H\left(\frac{F^{-1}(PD_{iT}) - \rho_i M_T}{\sqrt{1 - \rho_i^2}}\right) \tag{2}$$

where PD_{iT} is the unconditional probability of default (PD) that obligor i is in default at time T ; the cumulative distribution function (CDF) of Z_{iT} is denoted by $H(\cdot)$; and the CDF of V_{iT} is $F(\cdot)$, implying that the default threshold equals $F^{-1}(PD_{iT})$.

2.1.2. Obtaining the conditional loss

Under perfect granularity, the Law of Large Numbers implies that the conditional total loss on the portfolio, $TL|M$, is deterministic for any value of the common factor M

$$TL|M = \sum_i w_i \cdot E(LGD_i) \cdot H\left(\frac{F^{-1}(PD_i) - \rho_i M}{\sqrt{1 - \rho_i^2}}\right) \tag{3}$$

where time subscripts have been suppressed. In addition, the loss given default (LGD) of obligor i , LGD_i , is assumed to be independent of both the common and idiosyncratic factors.

2.1.3. Obtaining the unconditional loss with a given confidence level

The conditional total loss $TL|M$ is a decreasing function of the common factor M and, consequently, the unconditional distribution of TL can be derived directly on the basis of equation (3) and the CDF of the common factor, $G(\cdot)$. Denoting by $TL_{1-\alpha}$ the $(1-\alpha)^{th}$ percentile in the distribution of total losses, i.e., $\Pr(TL < TL_{1-\alpha}) = 1-\alpha$, it follows that

$$TL_{1-\alpha} = \sum_i w_i \cdot E(LGD_i) \cdot H\left(\frac{F^{-1}(PD_i) - \rho_i G^{-1}(\alpha)}{\sqrt{1 - \rho_i^2}}\right) = TL|M_\alpha \tag{4}$$

where $M_\alpha \equiv G^{-1}(\alpha)$ is the α^{th} percentile in the distribution of the common factor. The magnitude $TL_{1-\alpha}$ is also known as the credit Value-at-Risk (credit VaR) at the $(1-\alpha)$ confidence level.

2.1.4. Calculating the expected loss and unexpected loss

The expected loss (EL) of a portfolio is assumed to equal the proportion of obligors that might default within a given time frame, multiplied by the outstanding exposure at default (EAD), and once more multiplied by the LGD rate (i.e. the percentage of exposure that will not be recovered by sale of collateral). Mathematically, the EL of a portfolio can be written as follows

$$EL = \sum_i EAD_i \cdot E(LGD_i) \cdot PD_i \tag{5}$$

The capital requirement that covers unexpected loss (UL) (i.e. total minus expected), κ , on the entire portfolio with probability $(1 - \alpha)$ equals

$$\begin{aligned} \kappa &= TL_{1-\alpha} - \sum_i w_i \cdot E(LGD_i) \cdot PD_i \\ &= \sum_i w_i \cdot E(LGD_i) \cdot \left[H \left(\frac{F^{-1}(PD_i) - \rho_i G^{-1}(\alpha)}{\sqrt{1 - \rho_i^2}} \right) - PD_i \right] \equiv \sum_i w_i \cdot \kappa_i \end{aligned} \tag{6}$$

As implied by equation (6), the capital requirement for the portfolio can be set on the basis of exposure-specific parameters, which comprise the exposure's weight, as well as its LGD, PD and loading on the common factor. The flip side of this implication is that the portion of the capital requirement attributed to any particular exposure is independent of the rest of the portfolio and, thus, is portfolio invariant.

2.1.5. Expected shortfall

Artzner et al. (1997) proposed the use of expected shortfall (ES) (also called conditional VaR or tail VaR, which means excess loss beyond VaR) to alleviate the problems inherent in VaR. ES is the conditional expectation of loss given that the loss is beyond the VaR level at the $100(1 - \alpha)$ percent confidence level.

$$ES_\alpha(Loss) = E[Loss | Loss > VaR_\alpha(Loss)] \tag{7}$$

2.1.6. Distribution of asset values

The ASRF model assumes that the asset values are driven by the single common factor M and idiosyncratic factor Z . If the asset values are assumed to be standard normally distributed, we can re-write equation (1) as follows.

$$\begin{aligned} V_i &= \rho_i \cdot M + \sqrt{1 - \rho_i^2} \cdot Z_i; \quad Cov(Z_i, Z_j) = 0, \quad i \neq j; \quad Cov(M, Z_i) = 0, \quad \forall i; \tag{8} \\ M &\sim N(0,1), \quad Z_i \sim N(0,1), \forall i \end{aligned}$$

Apart from modeling default correlation through a normal or Gaussian copula, we have also considered the Student's t -distribution in our implementation.

2.1.7. Applying to the real world (including Basel II framework)

In practice, an implementation of the ASRF model requires that one specify the distribution of the common and idiosyncratic factors of asset returns. It is standard to assume normal distributions and rewrite equation (6) as

$$\kappa = \sum_i w_i \cdot E(LGD_i) \cdot \left[\Phi \left(\frac{\Phi^{-1}(PD_i) - \rho_i \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_i^2}} \right) - PD_i \right] \tag{9}$$

where $\Phi(\cdot)$ is the CDF of a standard normal variable. Equation (9) underpins the regulatory capital formula in the IRB approach of Basel II, in which $E(LGD_i) = 45\%$ and $\alpha = 0.1\%$.

2.2. Extensions

2.2.1. Random LGD

So far, we have assumed LGD to be constant. In our implementation, we also extend

the distribution of LGD to follow beta distribution.

2.2.2. Multi-state model

In multi-state modeling, we have considered the changes in credit quality along with their effects on the market value of the instruments in the portfolio. A straightforward way of modeling changes in credit quality is to assign borrowers to certain rating categories and allow transitions from one category to another by incorporating the transition probabilities migrating from one rating category to another.

2.3. Different implementation approaches

In our VBA implementation, we compute EL, VaR and ES by assuming different distributions of asset values and by means of different computational methods.

2.3.1. Analytical approach

Based on equation (4), we can calculate the $(1-\alpha)^{th}$ percentile in the distribution of total losses analytically. With specifications of PD, LGD, $F(\cdot)$ and $G(\cdot)$, we can compute the distribution of total losses of the credit portfolio.

2.3.2. Monte Carlo simulation

In Monte Carlo simulation, we start with just one systematic factor M . Asset values for each obligor in the portfolio are drawn randomly according to equation (8). For each obligor, we check whether there is a default for individual obligor through simulation runs. Obligor i defaults if its asset value falls below some threshold d_i chosen to match the specified PD_i : Default _{i} $\Leftrightarrow V_i \leq d_i$ and No default _{i} $\Leftrightarrow V_i > d_i$. We then aggregate the individual losses into a portfolio loss.

2.3.3. Importance sampling method

Simulating portfolio losses in the way we did in the previous section produces a lot of more or less irrelevant trials. In our implementation, we apply importance sampling method to adjust the simulation procedure of producing more relevant scenarios.

2.3.4. Quasi Monte Carlo method

We apply Quasi Monte Carlo method to generate a set of random numbers that have minimum deviation from the specified distribution with smaller number of simulation trials. In this paper, the Halton sequence with base 2 is used to generate a set of quasi-random numbers, which is in turn for simulating the distribution of portfolio.

3. The data

To simulate the real life situation, we constructed a sample bond fund portfolio which consists of 1,000 randomly selected bonds. According to Bloomberg, the credit rating of the bonds range from A to C as of 16 Jan 2013. We rely on the debt class in Table 1 to estimate the recovery rate. We map the 1-year default rate table and 1-year transition probability matrix in the *2012 Annual Global Corporate Default Study and Rating Transitions*² to individual bond of the sample bond fund portfolio. Given the PD, we specify the default correlation of individual bond by applying the Basel II risk weight formula³ without applying the size adjustment.

² <http://www.standardandpoors.com/ratings/articles/en/us/?articleType=HTML&assetID=1245348978068>

³ The formula is $\rho_i = 0.12 \times \left[\frac{1 - e^{-50 \times PD_i}}{1 - e^{-50}} \right] + 0.24 \times \left[1 - \frac{1 - e^{-50 \times PD_i}}{1 - e^{-50}} \right]$

Table 1: Recovery Rates

| Debt Class | Bank Loans | Senior secured | Senior unsecured | Senior subordinated | Subordinated | Junior subordinated |
|----------------------|------------|----------------|------------------|---------------------|--------------|---------------------|
| Recovery Rate | 84.5% | 65.7% | 49.3% | 36.8% | 26.1% | 13.6% |

Source: Van de Castle and Keisman (1999)

4. Results

Based on the sample portfolio and various combinations of model specification, we simulate the portfolio loss distribution and produce the simulation output in Table 2.

Table 2: Simulation Output

| Model | EL | VaR | | | ES | | |
|---|-------|-------|-------|--------|-------|-------|--------|
| | | 99% | 99.9% | 99.95% | 99% | 99.9% | 99.95% |
| Basel II standard formula | 1.09% | N/A | 2.58% | N/A | N/A | 2.81% | N/A |
| Analytical (normal) | 1.09% | 2.13% | 2.69% | 2.85% | 2.37% | 2.92% | 3.09% |
| Monte Carlo (normal) | 1.05% | 2.24% | 2.82% | 3.07% | 2.50% | 3.09% | 3.26% |
| Monte Carlo (normal) + importance sampling | 1.07% | 2.27% | 2.86% | 3.03% | 2.52% | 3.10% | 3.28% |
| Quasi Monte Carlo (normal) | 1.06% | 2.30% | 2.88% | 3.18% | 2.56% | 3.12% | 3.28% |
| Quasi Monte Carlo (normal) + importance sampling | 1.01% | 2.25% | 2.88% | 3.07% | 2.53% | 3.14% | 3.31% |
| Monte Carlo [student- <i>t</i> (df=20)] | 1.06% | 3.86% | 5.71% | 6.32% | 4.72% | 6.37% | 6.96% |
| Monte Carlo [student- <i>t</i> (df=20)] + importance sampling | 1.03% | 4.05% | 5.84% | 6.38% | 4.79% | 6.66% | 7.25% |
| Quasi Monte Carlo [student- <i>t</i> (df=20)] | 1.03% | 3.81% | 5.61% | 6.25% | 4.57% | 6.30% | 6.84% |
| Quasi Monte Carlo [student- <i>t</i> (df=20)] + importance sampling | 0.90% | 3.93% | 7.02% | 7.87% | 4.96% | 7.71% | 8.41% |
| Monte Carlo (normal) + Random LGD | 1.12% | 2.45% | 3.11% | 3.31% | 2.75% | 3.31% | 3.48% |
| Monte Carlo [student- <i>t</i> (df=20)] + Random LGD | 1.12% | 4.18% | 5.99% | 6.44% | 5.07% | 6.82% | 7.59% |
| Monte Carlo (normal) in multi-state | 1.06% | 2.72% | 3.51% | 3.88% | 3.04% | 3.87% | 4.18% |
| Monte Carlo [student- <i>t</i> (df=20)] in multi-state | 1.03% | 4.32% | 6.25% | 7.19% | 5.19% | 7.88% | 9.29% |
| Monte Carlo [student- <i>t</i> (df=20)] in multi-state + Random LGD | 1.12% | 4.79% | 7.50% | 8.43% | 5.95% | 8.42% | 8.95% |

Note: In our implementation, we set $\alpha = 0.01, 0.001$ and 0.0005 to calculate the VaR and ES with 99%, 99.9% and 99.95% confidence level.

5. Discussion

Under the Basel II IRB approach in which the LGD is set to 45%, the VaR under 99.9% confidence level of the sample portfolio is 2.58% of the portfolio value. This result is close to the VaR (2.69%) calculated under normal distribution assumption. The VaR substantially increases from 2.69% to 5.71% when the distribution is changed from normal to Student's *t*. It indicates the distributional assumption will affect the results significantly. When LGD is a risk factor in the model and multi-state model is implemented, the VaR increases to 5.99% and 6.25% respectively. The VaR of the model with both random LGD and multi-state effect is 7.5%. Based on these results, we found that distributional assumption is of prime importance in modeling of capital charge. Secondly, multi-state assumption is more important than random LGD. The VaR with *t*-distribution, random LGD and multi-state effect is three times of that under the Basel II IRB approach. For banks using Basel II IRB approach, the capital charge is substantially underestimated.

6. Conclusions

In this paper, we have reviewed the modeling framework of the ASRF model and assessed the portfolio credit risk by applying this model to a sample portfolio consists of 1,000 randomly selected bonds. We also studies the accuracy of credit risk measures and computation efficiency of various combinations of asset value assumptions (namely normal distribution vis-à-vis Student's t -distribution) and computation methods (namely, the analytical approach, standard Monte Carlo simulation, importance sampling and Quasi Monte Carlo simulation).

It is an industry practice to apply normality assumption for risk modeling. However, we found that the distributional assumption is crucial in modeling portfolio credit risk. For those banks who adopt normal distribution, constant LGD and single state as stipulated in IRB approach, it may bring about an oversimplification of the reality and an underestimation of the required capital charge.

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