On Decomposition of Point-symmetry in Square Contingency Tables

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Abstracts

For the analysis of square contingency tables, the issues of various symmetry rather than independence arise naturally. The point-symmetry (PS) model that indicates the structure of point-symmetry of cell probabilities and the marginal point-symmetry (MP) model that indicates the structure of point-symmetry of marginal probabilities are considered. The quasi point-symmetry (QP) model that indicates the structure of point-symmetry of odds ratios is also considered. For these models, the theorem that the PS model holds if and only if both the MP model and the QP model hold was shown. The purpose of present paper is to give a different proof of this result by using the minimum discrimination information (MDI) approach. Also, the MDI estimates of the cell frequencies of a square contingency table under hypotheses of some point-symmetry are given. Moreover, the associated MDI statistics are given, and the relationships between these test statistics are shown.

Keywords: marginal point-symmetry, minimum discrimination information, quasi point-symmetry

1. Introduction

Consider a contingency table which has same row and column classifications. For such the square contingency table, the issues of various symmetry rather than independence arise naturally. For example, the symmetry model indicates the symmetry of the cell probabilities, the quasi symmetry model indicates the symmetry of the odds ratios, and the marginal homogeneity model indicates the symmetry of the marginal probabilities. These models, that indicate the structure of symmetry with respect to the main diagonal of the table, are described in e.g., Agresti (2013, p. 426) and Bishop, Fienberg and Holland (1975, p. 282). Also, Caussinus (1966) showed that symmetry is equivalent to quasi symmetry and marginal homogeneity holding simultaneously.

Wall and Lienert (1976) considered the point-symmetry (PS) model that indicates the point-symmetry of the cell probabilities with respect to the center point (or cell) of the table. Tomizawa (1985) considered the quasi point-symmetry (QPS) model that indicates the point-symmetry of the odds ratios and the marginal point-symmetry (MPS) model that indicates the point-symmetry of the marginal probabilities (see Section 2) and showed that the PS model holds if and only if both the QPS model and the MPS model hold.

Ireland, Ku and Kullback (1969) proposed the method of estimation referred as the minimum discrimination information estimation (MDIE). They obtained the MDIEs for the symmetry model and the marginal homogeneity model and also gave a different proof of Caussinus' result as a consequence of MDI approach. We are now interested in considering the MDIEs under the hypotheses of some point-symmetry and in obtaining a different proof of Tomizawa's result by using the MDI approach.

The purposes of present paper are (i) to obtain the MDIEs for the PS model and the MPS model, (ii) to give a different proof of Tomizawa's result and (iii) to show the relationships between test statistics. Also in the last section we prove that under certain conditions, the QPS model is the closet model to PS when distance is measured by the Kullback-Leibler distance.

2. Models

For an $r \times r$ square contingency table with same row and column classifications, let p_{ij} denote the probability that an observation will fall in the ith row and the jth column of the table ($i=1,\ldots,r;\ j=1,\ldots,r$). Wall and Lienert (1976) considered the PS model defined by

$$p_{ij} = p_{i^*j^*} \quad (i=1,\ldots,r;\ j=1,\ldots,r), \tag{2.1}$$

where the symbol "*" denotes $i^* = r+1-i$. This model indicates a structure of point-symmetry of the cell probabilities with respect to the center point (when r is even) or the center cell (when r is odd) in a square table. Tomizawa (1985) considered the QPS and MPS models. The QPS model is defined by

$$p_{ij} = \mu\alpha_i\beta_j\psi_{ij} \quad (i=1,\ldots,r;\ j=1,\ldots,r), \tag{2.2}$$

where $\psi_{ij} = \psi_{i^*j^*}$. The QPS model can be expressed as

$$\theta_{(i<j;s<t)} = \theta_{(j^*<i^*;t^*<s^*)} \quad (i<j; s<t),$$

where $\theta_{(i<j;s<t)} = (p_{is}p_{jt})/(p_{js}p_{it})$. Therefore the QPS model has its characterization in terms of point-symmetry of odds ratios. The MPS model is defined by

$$p_{i+} = p_{i^*+} \ \text{ and } \ p_{+i} = p_{+i^*} \quad (i=1,\ldots,r), \tag{2.3}$$

where $p_{i+} = \sum_{t=1}^{r} p_{it}$ and $p_{+i} = \sum_{s=1}^{r} p_{si}$. This indicates that the row (column) marginal distributions are point symmetric with respect to the midpoint of the row (column) categories.

Tomizawa (1985) also gave the decomposition of the PS model such that the PS model holds if and only if both the QPS model and the MPS model hold.

3. MDIEs for PS and MPS

Ireland et al. (1969) proposed the method of estimation referred as MDIE (see also Bishop et al., 1975, p. 346; Read and Cressie, 1988, p. 34). In this section, we give MDIEs for PS and MPS.

Let n_{ij} denote the observed frequency of the (i,j)th cell in a table and let $\pi_{ij} = n_{ij}/n$ where $n = \sum\sum n_{ij}$ is the size of multinomial sample. The $\{p_{ij}\}$ which minimize the discrimination information

$$I(p:\pi) = \sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij} \log \frac{p_{ij}}{\pi_{ij}},$$

subject to the null hypothesis of PS i.e., equation (2.1) may be obtained by minimizing

$$\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij} \log \frac{p_{ij}}{\pi_{ij}} + \sum_{D_1} \lambda_{ij}(p_{ij} - p_{i^*j^*}) + \phi\left(\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij} - 1\right),$$

with respect to $\{p_{ij}\}$ where $\{\lambda_{ij}\}$ and ϕ are undermined Lagrangian multipliers and

$$D_1 = \begin{cases} \left\{(i,j) \mid i=1,\ldots,\frac{r}{2};\ j=1,\ldots,r\right\} & (r \textit{ is even}), \\ \left\{(i,j) \mid i=1,\ldots,\frac{r-1}{2};\ j=1,\ldots,r \textit{ and } i=\frac{r+1}{2};\ j=1,\ldots,\frac{r-1}{2}\right\} & (r \textit{ is odd}). \end{cases}$$

Thus the MDIEs for the PS model are given by

$$p_{ij}^{PS} = d(\pi_{ij}\pi_{i^*j^*})^{\frac{1}{2}} \quad (i=1,\ldots,r;\ j=1,\ldots,r),$$

where $d = 1/\sum\sum(\pi_{ij}\pi_{i^*j^*})^{\frac{1}{2}}$. Also, we see that the minimum value of $I(p:\pi)$ is $\log d$.

As a similar manner to the case of the PS model, the MDIEs for the MPS model i.e., equation (2.3) may be obtained by minimizing

$$\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}\log\frac{p_{ij}}{\pi_{ij}}+\sum_{i=1}^{[r/2]} a_i(p_{i+}-p_{i^*+})+\sum_{i=1}^{[r/2]} b_i(p_{+i}-p_{+i^*})+\phi\left(\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij}-1\right),$$

with respect to $\{p_{ij}\}$ where $\{a_i\}$, $\{b_i\}$ and ϕ are undermined Lagrangian multipliers and $[k]$ is the greatest integer that is less than or equal to k. It may be shown that the minimizing values are given by

$$p_{ij}^{MPS}=c\pi_{ij}\exp[-(A_i+B_j)] \quad (i=1,\dots,r;\ j=1,\dots,r), \tag{3.1}$$

where $c=1/\sum\sum\pi_{ij}\exp[-(A_i+B_j)]$,

$$A_i=\begin{cases} a_i & (1\le i\le \frac{r}{2}),\\ -a_i & (\frac{r}{2}+1\le i\le r),\end{cases} \qquad B_j=\begin{cases} b_j & (1\le j\le \frac{r}{2}),\\ -b_j & (\frac{r}{2}+1\le j\le r),\end{cases}$$

when r is even, and

$$A_i=\begin{cases} a_i & (1\le i\le \frac{r-1}{2}),\\ 0 & (i=\frac{r+1}{2}),\\ -a_i & (\frac{r+3}{2}\le i\le r),\end{cases} \qquad B_j=\begin{cases} b_j & (1\le j\le \frac{r-1}{2}),\\ 0 & (j=\frac{r+1}{2}),\\ -b_j & (\frac{r+3}{2}\le j\le r),\end{cases}$$

when r is odd. Also the minimum value of $I(p:\pi)$ is $\log c$. The p_{ij}^{MPS} in (3.1) may be determined by using e.g., Newton-Raphson method.

4. Relationships between some point-symmetry

Tomizawa (1985) gave the theorem that the PS model holds if and only if both the QPS model and the MPS model hold. We shall show a different proof by using the MDI approach.

We have

$$\sum_i\sum_j p_{ij}^{PS}\log\frac{p_{ij}^{PS}}{\pi_{ij}}=\sum_i\sum_j p_{ij}^{PS}\log\frac{p_{ij}^{MPS}}{\pi_{ij}}+\sum_i\sum_j p_{ij}^{PS}\log\frac{p_{ij}^{PS}}{p_{ij}^{MPS}},$$

and using p_{ij}^{MPS} in equation (3.1)

$$\sum_i\sum_j p_{ij}^{PS}\log\frac{p_{ij}^{MPS}}{\pi_{ij}}=\log c=\sum_i\sum_j p_{ij}^{MPS}\log\frac{p_{ij}^{MPS}}{\pi_{ij}}.$$

Thus we can get

$$I(p^{PS}:\pi)=I(p^{MPS}:\pi)+I(p^{PS}:p^{MPS}). \tag{4.1}$$

As a similar manner to Section 3, we derive the minimum of $I(p:p^{MPS})$ subject to the hypothesis of PS. We have

$$\tilde{p}_{ij}^{PS}=k(p_{ij}^{MPS}p_{i^*j^*}^{MPS})^{\frac{1}{2}} \quad (i=1,\dots,r;\ j=1,\dots,r),$$

where $k=1/\sum\sum(p_{ij}^{MPS}p_{i^*j^*}^{MPS})^{\frac{1}{2}}$. Now using the MDIEs for MPS in equation (3.1) we have

$$\tilde{p}_{ij}^{PS}=kc(\pi_{ij}\pi_{i^*j^*})^{\frac{1}{2}} \quad (i=1,\dots,r;\ j=1,\dots,r),$$

where $kc = 1/\sum\sum(\pi_{ij}\pi_{i^*j^*})^{\frac{1}{2}}$. So we can see $\tilde{p}_{ij}^{PS} = p_{ij}^{PS}$ for all (i, j) and $kc = d$. Thus the equation (4.1) may be written as $\log d = \log c + \log k$.

When the PS model holds, we obviously see that both the QPS model and the MPS model hold. So we shall show the PS model hold assuming that both the QPS model and the MPS model hold. If q has MPS, then $I(p^{MPS} : q) = 0$ and $p^{MPS} = q$. Moreover, since the QPS model i.e., $q_{ij} = \mu\alpha_i\beta_j\psi_{ij}$ where $\psi_{ij} = \psi_{i^*j^*}$ (equation (2.2)) holds, we have

$$I(p^{PS} : q) = I(p^{PS} : p^{MPS}) = \log d + \sum_i\sum_j p_{ij}^{PS} \log\left(\frac{\alpha_{i^*}\beta_{j^*}}{\alpha_i\beta_j}\right)^{\frac{1}{2}},$$

from equation (4.1). Also, we have

$$\sum_i\sum_j p_{ij}^{PS} \log\left(\frac{\alpha_{i^*}\beta_{j^*}}{\alpha_i\beta_j}\right)^{\frac{1}{2}} = 0.$$

Therefore we can see

$$I(p^{PS} : q) = \log d \geq 0. \tag{4.2}$$

On the other hand, since q satisfies the structure of both QPS and MPS by assumption, we have

$$I(q : p^{PS}) = -\log d \geq 0. \tag{4.3}$$

From equations (4.2) and (4.3) we get

$$I(p^{PS} : q) = 0,$$

which implies $p^{PS} = q$. Namely q satisfies the structure of PS.

We also consider the MDI statistics (MDISs) for PS and MPS. By using the MDIEs for PS, we can obtain the associated MDIS for PS as follows:

$$2nI(p^{PS} : \pi) = 2n\log d,$$

where $\pi_{ij} = n_{ij}/n$ with $n = \sum\sum n_{ij}$. This is asymptotically distributed as chi-square distribution with $r^2/2$ (when r is even) or $(r^2-1)/2$ (when r is odd) degrees of freedom under the null hypothesis of PS.

Also we can get the MDIS for MPS as follows:

$$2nI(p^{MPS} : \pi) = 2n\log c,$$

where $\pi_{ij} = n_{ij}/n$ with $n = \sum\sum n_{ij}$. This is asymptotically distributed as chi-square distribution with r (when r is even) or $r-1$ (when r is odd) degrees of freedom under the null hypothesis of MPS.

From equation (4.1), we can obtain a result for the MDISs as follows:

$$2nI(p^{PS} : \pi) = 2nI(p^{MPS} : \pi) + 2nI(p^{PS} : p^{MPS}),$$

where $\pi_{ij} = n_{ij}/n$ with $n = \sum\sum n_{ij}$. We note that $2nI(p^{PS} : p^{MPS})$ is asymptotically distributed as chi-square distribution with $((r-1)^2-1)/2$ (when r is even) or $(r-1)^2/2$ (when r is odd) degrees of freedom under the null hypothesis of QPS. Note that Tahata and Tomizawa (2008) discussed about the relationships between test statistics for the hypotheses of PS, MPS and QPS.

5. Discussions

Bishop et al. (1975, p. 347) discussed about the MDIEs for symmetry. (Note that they referred to MDIEs as modified MDIEs.) When the diagonal cells are included in the analysis, we get estimated expected frequencies greater than the corresponding observed frequencies for the main diagonal cells. They mentioned that such estimates are intuitively unappealing. Same problem occurs for point-symmetry when r is odd. Since the geometric mean of two unequal numbers is less than the arithmetic mean, we have

$$(n_{ij}n_{i^*j^*})^{\frac{1}{2}} < \frac{n_{ij}+n_{i^*j^*}}{2},$$

unless $n_{ij} = n_{i^*j^*}$. Thus using p_{ij}^{PS} in Section 3 we see

$$\begin{aligned}\sum_i\sum_j np_{ij}^{PS} &= d\left(\sum\sum_{(i,j)\neq(c,c)}\left(n_{ij}n_{i^*j^*}\right)^{\frac{1}{2}} + n_{cc}\right),\\ &< d\left(\sum\sum_{(i,j)\neq(c,c)}\frac{n_{ij}+n_{i^*j^*}}{2} + n_{cc}\right),\\ &= dn,\end{aligned}$$

where $c=(r+1)/2$, and d must be greater than 1. When the center cell is included in the analysis, we get cell estimate greater than the observed count for the center cell. On the other hand, when r is even such problem does not occur. Therefore, the MDIEs for PS may be appealing when r is even rather than odd.

As a similar manner to Kateri and Papaioannou (1997), we shall consider another interpretation of QPS. Assume that the row and column marginals are given. The problem is to minimize $I(p:q^{PS})$ for $q_{ij}^{PS} = (p_{ij} + p_{i^*j^*})/2$ ($i=1,\ldots,r;\, j=1,\ldots,r$) under the restrictions

$$p_{i+} = a_i \quad \text{and} \quad p_{+i} = b_i,$$

for $i=1,\ldots,r$ and

$$p_{ij} + p_{i^*j^*} = 2q_{ij}^{PS} \quad (i=1,\ldots,r;\, j=1,\ldots,r),$$

where a_i and b_i are the given marginal with $\sum a_i = \sum b_i = 1$ and q_{ij}^{PS} are the cell probabilities under PS.

This is a constraint minimization problem, which can be solved by the method of Lagrange multipliers. The Lagrange function to be minimized is

$$\begin{aligned}&\sum_i\sum_j p_{ij}\log\frac{p_{ij}}{q_{ij}^{PS}} + \sum_i \lambda_{1(i)}(p_{i+}-a_i)\\ &\quad + \sum_i \lambda_{2(i)}(p_{+i}-b_i) + \sum_i\sum_j \lambda_{12(ij)}(p_{ij}+p_{i^*j^*}-2q_{ij}^{PS}).\end{aligned} \tag{5.1}$$

Equating to 0 the derivative of equation (5.1) with respect to p_{ij}, we obtain

$$\log\frac{p_{ij}}{q_{ij}^{PS}} + 1 + \lambda_{1(i)} + \lambda_{2(j)} + \lambda_{12(ij)} + \lambda_{12(i^*j^*)} = 0. \tag{5.2}$$

Thus equation (5.2) leads to

$$p_{ij} = q_{ij}^{PS} e^{-1-\lambda_{1(i)}-\lambda_{2(j)}+\bar{\lambda}_{12(ij)}}, \tag{5.3}$$

where $\bar{\lambda}_{12(ij)} = -\lambda_{12(ij)} - \lambda_{12(i^*j^*)}$. Since

$$p_{ij} + p_{i^*j^*} = q_{ij}^{PS} e^{\bar{\lambda}_{12(ij)}-1}(\alpha_i\beta_j + \alpha_{i^*}\beta_{j^*}),$$

where $\alpha_i = e^{-\lambda_{1(i)}}$ and $\beta_j = e^{-\lambda_{2(j)}}$ from equation (5.3), we obtain

$$e^{\bar{\lambda}_{12(ij)}-1} = \frac{2}{\alpha_i\beta_j + \alpha_{i^*}\beta_{j^*}}.$$

Thus we get

$$\frac{p_{ij}}{p_{ij}+p_{i^*j^*}} = \frac{\alpha_i\beta_j}{\alpha_i\beta_j + \alpha_{i^*}\beta_{j^*}}. \tag{5.4}$$

The equation (5.4) is equivalent to the equation (2.2). Also since the function

$f(x) = x \log x$ is a strictly convex function (the second derivative of it is positive), the Hessian matrix is positive definite. Thus the fact ensures that equation (5.1) has a minimum at p_{ij}. Therefore, we see that in the class of models with given row (a_i) and column (b_i) marginals ($i = 1, \ldots, r$) and with given sums $p_{ij} + p_{i^* j^*} = 2q_{ij}^{PS}$ ($i = 1, \ldots, r; j = 1, \ldots, r$), the QPS model is the model closest to the PS model in terms of the Kullback-Leibler distance.

Tomizawa (1985) considered the point-symmetry model for an $r \times c$ contingency table. Also he showed that the point-symmetry model holds if and only if both the quasi point-symmetry model and the marginal point-symmetry model for the $r \times c$ contingency table. The results given in this paper could be extended for the $r \times c$ contingency tables.

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