On Phase II nonparametric CUSUM exceedance control charts for unknown location

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Abstract

Standard control charts are often based on the assumption that the observations follow a specific parametric distribution, such as the normal. In many applications we do not have enough information to make this assumption and in such situations, development and application of control charts that do not depend on a particular distributional assumption is desirable. Nonparametric or distribution-free control charts can serve this wider purpose. A key advantage of nonparametric charts is that the in-control run-length distribution is the same for all continuous process distributions. In this paper we examine several aspects related to the efficient design and implementation of a class of Phase II nonparametric cumulative sum (CUSUM) charts based on the exceedance statistic. Here we investigate which order statistic (percentile), from the reference (Phase I) sample, should be used to obtain the best performance. It is observed that other choices than the median, such as the third quartile, can play an important role in improving the performance of the chart. We also study different choices of the CUSUM design parameter, *k,* called the reference value. Moreover, although the most widely used chart performance metric is the average run-length (*ARL*), certain shortcomings have been observed and instead we use more representative measures for the assessment of chart performance. These include other percentiles of the run-length, more specifically, the median run-length (*MRL*), which provides additional and more meaningful information about the in-control and out-of-control performances of control charts, not given by the *ARL*. The procedures are illustrated with some data. A summary and some concluding remarks are given.

Keywords: Distribution-free, median run-length, order statistics, robustness

1. Introduction

Control charts are effective tools in Statistical Process Control (SPC) for monitoring a process over time. Charts that are constructed on the assumption of a specific form of a parametric distribution are called parametric control charts. In many applications, however, there is not enough information to justify this assumption and control charts that do not require a particular distributional assumption are desirable. Nonparametric or distribution-free control charts can serve this broader purpose. A key advantage of nonparametric charts is that their in-control (IC) run-length distribution is the same for all continuous process distributions. This is not true for parametric control charts in general and consequently their IC robustness can be a legitimate concern. Moreover, nonparametric charts are often more robust and efficient under non-normal distributions. For a thorough account of the nonparametric control charts literature see Chakraborti et al. (2001), Chakraborti and Graham (2007) and Chakraborti et al. (2011). While the Shewhart-type charts are the most widely known charts in practice because of their simplicity and global performance, other classes of charts, such as the cumulative sum (CUSUM) charts are useful and sometimes more naturally appropriate in the process control environment in view of the sequential nature of data collection. These charts are based on the cumulative totals of a suitable statistic, obtained as the data accumulate, and are known to be more efficient for detecting persistent shifts in the process. In this paper we study a class of Phase II nonparametric cumulative sum (denoted NPCUSUM) charts for monitoring the unknown location parameter, based on the exceedance statistic. The key focus in this paper is investigating which order statistic, from the Phase I sample, should be used for good performance. Note that the NPCUSUM chart using the reference sample median was first introduced and studied by Mukherjee et al. (2013), and is referred to as the nonparametric exceedance CUSUM (denoted NPCUSUM-EX) chart. For comparison purposes, we include a recent nonparametric chart, the NPCUSUM chart based on the Wilcoxon rank-sum statistic (denoted NPCUSUM-Rank), proposed by Li et al. (2010).

2. Statistical framework and preliminaries of CUSUM charts

Several NPCUSUM charts have been developed: For monitoring the specified location of a continuous process see e.g. McGilchrist and Woodyer (1975), Bakir and Reynolds (1979) and Amin et al. (1995). On the other hand, for monitoring the unknown location of a process see e.g. McDonald (1990), Jones et al. (2004) and Li et al. (2010). These are examples of univariate Phase II NPCUSUM control charts that are useful when the location is unknown and charting is based on an IC reference (Phase I) sample. Assume that a reference sample $X_1, X_2, \dots, X_m$ is available from an IC process with an unknown continuous cdf $F(x)$. Let $Y_{j1}, Y_{j2}, \dots, Y_{jn_j}$, $j = 1,2,\dots$, denote the j^{th} test Phase II sample of sizen_j. Let $G(y)$ denote the cdf of the distribution of the j^{th} Phase II sample. Both F and G are unknown continuous cdf's and the process is IC when $F = G$. For detecting a change in the location, we use the location model $G_Y(x) = F(x - \theta)$ where $\theta \epsilon (-\infty, \infty)$ is the unknown location parameter so the process is IC when $\theta = 0$. It is often the case that the Phase II samples (subgroups) are all of the same size, n, so that the subscript j can be suppressed.

2.1 NPCUSUM-EX control chart

Let $U_{j,r}$ denote the the number of Y observations in the j^{th} Phase II sample that exceeds $X_{(r)}$, the r^{th} ordered observation in the reference sample. The statistic $U_{j,r}$ is called an exceedance statistic and the probability $p_r = P[Y > X_{(r)} | X_{(r)}]$ is called an exceedance probability. Mukherjee et al. (2013) proposed the NPCUSUM-EX chart which may be introduced as follows. Given $X_{(r)} = x_{(r)}$, it can be shown (see Result A.1 in the Appendix of Mukherjee et al. (2013)) that the $U_{j,r}$ follows a binomial distribution with parameters (n, p_r) and thus, conditionally on $X_{(r)}$, we can use a binomial-type CUSUM chart based on the $U_{j,r}$'s to monitor the process location. Noting that $E\left(U_{j,r} \middle| X_{(r)}\right) = np_r$ and the conditional probability p_r is unknown, we may replace it by its unconditional IC value $d = \frac{m - r + 1}{m + 1}$ (the reader is referred to Result A.4 in the Appendix of Mukherjee et al. (2013) for the derivation of d). Hence the two-sided NPCUSUM-EX chart has plotting statistics

$$C_j^+ = \max[0, C_{j-1}^+ + \left(U_{j,r} - nd\right) - k] \text{ and } C_j^- = \min[0, C_{j-1}^- + \left(U_{j,r} - nd\right) + k]$$

for $j = 1,2,3,\dots$ with starting values $C_0^+ = C_0^- = 0$ and $k \geq 0$ is the reference value. The chart signals a possible OOC situation for the first j at which either $C_j^- < -H$ or $C_j^+ > H$, where $H > 0$ is called a decision constant. While constructing their NPCUSUM-EX chart, Mukherjee et al. (2013) focused on the median of the reference sample with the reasoning that the median is a robust and good representative of the reference data and one of the popular percentiles in practice. They were also swayed by earlier research in the fields of nonparametric hypothesis testing and control charts. However, the question of performance of the chart, depending on which reference sample order statistic is chosen to create the chart, has not been examined. To this end, in this paper we investigate the performance of the NPCUSUM-EX chart systematically, based on the 25th, 40th, 50th, 60th and 75th percentile, respectively.

2.2 NPCUSUM-Rank control chart

Li et al. (2010) recently proposed a NPCUSUM chart for location based on the well-known Wilcoxon rank-sum with the following plotting statistics

$$C_j^+ = \max\left[0, C_{j-1}^+ + \left(W_j - \tfrac{n(m+n+1)}{2}\right) - k\right] \text{ and } C_j^- = \min\left[0, C_{j-1}^- + \left(W_j - \tfrac{n(m+n+1)}{2}\right) + k\right]$$

for $j = 1,2,3, \ldots$ where the starting values $C_0^+ = C_0^- = 0$ and $k \geq 0$ is the reference value. The chart signals a possible OOC situation for the first j at which either $C_j^- < -H$ or $C_j^+ > H,$ where $H > 0$. Otherwise, the process is considered IC and process monitoring continues without interruption.

3. Implementation and performance

The most widely used chart performance metric is the average run-length (*ARL*) and determining the charting constants typically involve specifying a nominal IC *ARL* value, such as 500. However, since the run-length distribution is significantly right-skewed, researchers have advocated using other, more representative, measures for the assessment of chart performance. These include other percentiles of the run-length, more specifically, the median run-length (*MRL*), which provides more meaningful information about the IC and OOC performances of control charts, not given by the *ARL*. The median is far less sensitive to measurement errors and the median is a more robust measure, in that it is far less impacted by outliers.

3.1 Implementation: Choice of chart design parameters

The design parameters k and H are chosen so that the chart has a specified nominal MRL_0. The first step in this direction is to choose k. Under the normal distribution, the choice of k has been discussed by numerous authors, see, for example, Lucas (1985) and Montgomery (2009). Lucas (1985) stated "The CUSUM parameter k is determined by the acceptable mean level (μ_a) and by the unacceptable mean (μ_d) level which the CUSUM scheme is to detect quickly. For normally distributed variables the k value is chosen half way between the acceptable mean level and the unacceptable mean level." In more recent literature see e.g. Montgomery (2009), it is agreed that in the normal theory setting k is typically chosen relative to the size of the shift we want to detect, that is, $k = \frac{1}{2}\delta$, where δ is the size of the shift in standard deviation units. In this study we set the reference value equal to $k = \delta STDEV(U_{j,r})$ and $k = \delta STDEV(W_j)$ for the NPCUSUM-EX and the NPCUSUM-Rank charts, respectively, with δ = 0.00(0.25)1.00, 1.50, 2.00 and 3.00. Note that although shifts as large as δ = 3.00 were considered in this study, the largest magnitude reported in the paper is δ = 1.00, since, for larger shifts, the run-length characteristics of the charts tend to converge. After choosing k, the next step is to choose H, in conjunction with the chosen k, so that a desired nominal MRL_0 is attained. However, for a discrete random variable the chances are that H cannot always be found such that the desired nominal MRL_0, say 350, is attained exactly and hence using a conservative approach, H is found so that the attained MRL_0 is less than or equal to the desired nominal MRL_0. The decision interval, H, is found using a search algorithm using 100,000 Monte-Carlo simulations.

3.2 Out-of-control control chart performance comparisons

Note that *SymmMixN* denotes the Symmetric Mixture Normal distribution [$0.6N(\mu_1 = 0,\ \sigma_1 = 0.25) + 0.4N(\mu_2 = 0,\ \sigma_2 = 4)$], *AsymmMixN1* denotes the Asymmetric Mixture Normal distribution [$0.6N(\mu_1 = 1,\ \sigma_1 = 1) + 0.4N(\mu_2 = 0,\ \sigma_2 = 4)$] and *AsymmMixN2* denotes the Asymmetric Mixture Normal distribution [$0.6N(\mu_1 = -1, \sigma_1 = 1) + 0.4N(\mu_2 = 0, \sigma_2 = 4)$].

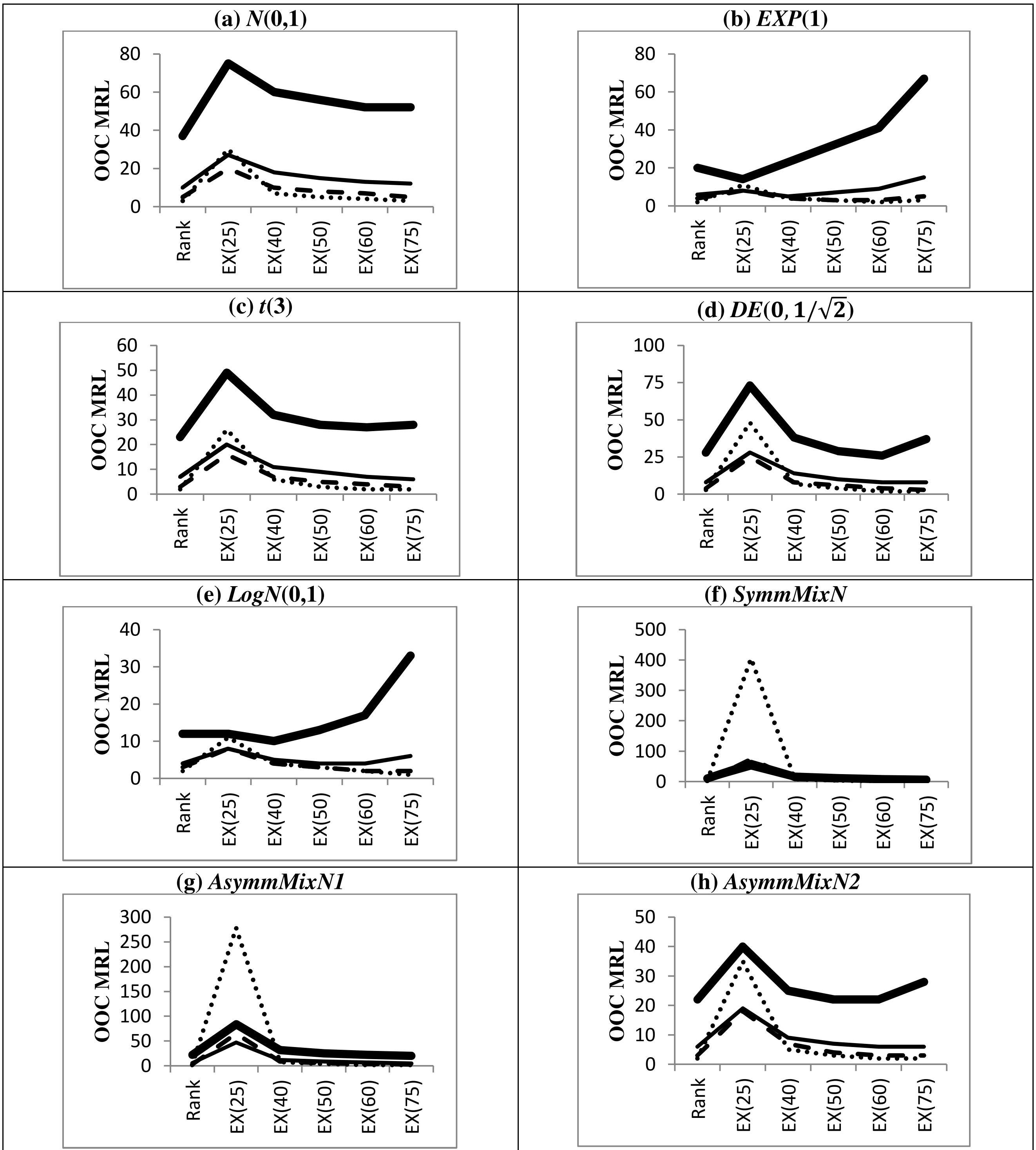

Figure 1. OOC performance comparison of the NPCUSUM charts across various distributions for $m = 100$ and $n = 5$

A summary of the figures and some recommendations are listed below. Note that shorthand notation is used to describe the charts, for example, the CUSUM-EX chart based on the 50th percentile is denoted by EX(50) etc., and the CUSUM-Rank chart is denoted by Rank.

<table>
<tr><td>Distribution</td><td>Small shifts ($\delta \leq 0.50$)</td><td>Medium to large shifts ($\delta > 0.50$)</td></tr>
<tr><td>N(0,1)</td><td>The Rank chart is better than EX chart.</td><td rowspan="6">The EX(60 or 75) chart is the best for all distributions. In addition, as the size of the shift increases the exceedance chart based on the 75th percentile performs the best and almost immediately detects the shift. The recommendation is to use the exceedance chart based on the 75th reference percentile which will signal almost immediately.</td></tr>
<tr><td>EXP(1)</td><td>The EX(25) chart is better than the Rank chart for $\delta = 0.25$ and the EX(40) chart is better than the Rank chart for $\delta = 0.50$.</td></tr>
<tr><td>t(3)</td><td>Although the Rank chart is performing the best for $\delta = 0.25$, the EX(75) chart is performing the best for $\delta = 0.50$.</td></tr>
<tr><td>$DE(0,1/\sqrt{2})$</td><td>The EX(60) chart is better than the Rank chart for $\delta = 0.25$.
For $\delta = 0.50$, the EX(60 or 75) chart and the Rank chart perform similarly.</td></tr>
<tr><td>LogN(0,1)</td><td>For $\delta = 0.25$ the EX(40) chart is best, whereas for $\delta = 0.50$ the EX(50 or 60) chart and the Rank chart perform the best.</td></tr>
<tr><td>SymmMixN
AsymmMixN1
and
AsymmMixN2</td><td>EX(60 or 75) chart is the best</td></tr>
</table>

4. Summary and concluding remarks

CUSUM charts are popular control charts used in practice; they take advantage of the sequential accumulation of data arising in a typical SPC environment and are known to be more efficient than Shewhart charts in detecting smaller and persistent shifts. However, the traditional (parametric) CUSUM charts can lack IC robustness and as such the possibility of varying and unknown false alarm rates is a practical concern with their applications. Nonparametric CUSUM (denoted NPCUSUM) charts offer an attractive alternative as they combine the inherent advantages of nonparametric charts (IC robustness; same, fixed, false alarm rate for all continuous distributions) with the better small shift detection capability of CUSUM-type charts. We examine a class of NPCUSUM charts based on the exceedance statistic by investigating which order statistic (percentile), from the reference sample, should be used for good overall performance. For comparison purposes an alternative rank-based nonparametric chart, the NPCUSUM chart based on the Wilcoxon rank-sum statistic, is used and it is seen that the exceedance CUSUM chart based on higher percentiles performs better than its competitors in many cases for a number of distributions. More specifically, it is seen that for moderate to large shifts there is little doubt that the practitioner should use the exceedance chart based on the 75th percentile which will signal almost immediately.

References

Amin, R.W., Reynolds, Jr. M.R. and Bakir, S.T. (1995). Nonparametric quality control charts based on the sign statistic. *Communications in Statistics: Theory and Methods,* 24 (6), 1597-1623.

Bakir, S.T. and Reynolds, Jr. M.R. (1979). A nonparametric procedure for process control based on within-group ranking. *Technometrics*, 21 (2), 175-183.

Chakraborti, S. and Graham, M.A. (2007). Nonparametric control charts. *Encyclopedia of Statistics in Quality and Reliability,* 1, 415 – 429, John Wiley, New York.

Chakraborti, S., Human, S.W. and Graham, M.A. (2011). Nonparametric (distribution-free) quality control charts. In Handbook of Methods and Applications of Statistics: Engineering, Quality Control, and Physical Sciences. N. Balakrishnan, Ed., 298-329, John Wiley & Sons, New York.

Chakraborti, S., Van der Laan, P. and Bakir, S.T. (2001). Nonparametric control charts: An overview and some results. *Journal of Quality Technology,* 33 (3), 304-315.

Chakraborti, S., Van der Laan, P. and Van de Wiel, M.A. (2004). A class of distribution-free control charts. *Journal of the Royal Statistical Society. Series C*: *Applied Statistics*, 53 (3), 443-462.

Jones, L. A., Champ, C. W. and Rigdon, S.E. (2004). The run length distribution of the CUSUM with estimated parameters. *Journal of Quality Technology*, 36 (1), 95-108.

Li, S.Y., Tang, L.C. and Ng, S.H. (2010). Nonparametric CUSUM and EWMA control charts for detecting mean shifts. *Journal of Quality Technology*, 42 (2), 209-226.

Lucas, J.M. (1985). Cumulative sum (CUSUM) control schemes. *Communications in Statistics: Theory and Methods,* 14, 2689-2704.

McDonald, D. (1990). A CUSUM procedure based on sequential ranks. *Naval Research Logistics*, 37 (5), 627-646.

McGilchrist, C.A. and Woodyer, K.D. (1975). Note on a distribution-free CUSUM technique. *Technometrics,* 17 (3), 321-325.

Montgomery, D.C. (2009). *Statistical Quality Control: A Modern Introduction*, 6th ed., John Wiley, New York, NY.

Mukherjee, A., Graham, M.A. and Chakraborti, S. (2013). "Distribution-free exceedance CUSUM control charts for location." *Communications in Statistics - Simulation and Computation*, 42 (5), 1153-1187.