

Mills Ratios of Skew-t Distribution and Their Applications

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Abstract

Skew distributions such as skew-normal and skew-t distributions have been widely received attentions in both theoretical and applied statistics especially in economics, insurance and finance. Empirical analysis shows that the skew-t distribution family is a reasonable alternative in modeling datasets with skewness, kurtosis and extreme tail. It is well-known in extreme value theory that the asymptotic behavior of distributional tail can determine which domain of attractions the distribution belongs to. In this paper, we consider the distributional tail behaviors of skew-t distribution and the Mills type inequality and Mills type ratio of skew-t distribution are derived. Two applications are also considered: One is to derive the asymptotic distributions of extremes of a sequence of independent and identically distributed random variables obeying the skew-t distribution, which shows that skew-t distribution belongs to the domain of attraction of Frchet extreme value distribution, i.e., the limiting distribution of normalized skew-t extremes is Frchet distribution. The other application is for the approximation of large quantiles of skew-t distribution by using the Mills inequalities. Numerical results are also demonstrated to study the performance of the proposed approximation of large quantiles.

Keywords: Tail behavior; Extreme value distribution; Large quantile.

1. Introduction

With favorable properties on skewness, kurtosis and heavy tailedness, skew-t distribution has been widely applied in economics, insurance and finance. A random variable X is said to have a standard skew-t distribution with shape parameter $\lambda \in R$ (written as $X \sim ST(\nu, \lambda)$) if its probability density function (pdf) is

$$f_{\nu,\lambda}(x) = 2t_{\nu}(x)T_{\nu}(\lambda x), \quad -\infty < x < +\infty, \quad (1)$$

where $\lambda > 0$ is the shape parameter, $t_{\nu}(\cdot)$ and $T_{\nu}(\cdot)$ denote the pdf and cumulative distribution function (cdf) of standard Student- t distribution respectively. Let $F_{\nu,\lambda}(x)$ denote the cdf corresponding to (1). Obviously $ST(\nu, 0)$ is a standard Student- t distribution.

It is well-known in extreme value theory that the asymptotic distributional tail decides which max-domain of attractions the distribution belongs to, for more details see Resnick (1987). For a given distribution function, Mills type ratio of the distribution is one keystone to test which max-domain of attractions the distribution belongs to and to find the suitable norming constants, see Resnick (1987) and Leadbetter et al. (1983). For normal distribution, Mills ratio and Mills inequality were studied by Mills (1926), Birnbaum (1942), Sampford (1953) and Boyd (1959). The following Mills inequality and Mills ratio of Student- t distributions are due to Soms (1976, 1980):

$$\frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) \left(1 - \frac{\nu}{(\nu + 2)x^2} \right) < \frac{1 - T_{\nu}(x)}{t_{\nu}(x)} < \frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) \quad (2)$$

for all $x > 0, \nu > 0$, and

$$\frac{1 - T_\nu(x)}{t_\nu(x)} \sim \frac{x}{\nu} \tag{3}$$

as $x \rightarrow \infty$.

Mills inequality and Mills ratio are also useful to calculate large quantiles. Reiss (1989) derived an approximation formula for high Gaussian quantiles. For student-t distribution, Gafer and Kafader (1984) gave a general formula for all quantiles. Schlüter and Fischer (2012) derived an analytic approximation for student-t tail quantiles by using generalized Reiss' procedure and the fact that the Student-t distribution arises as the limit of a variance-mixture of normals. To the best of our knowledge, only Liao et al. (2012) considered the Mills inequalities and Mills ratios of skew-normal distribution, and there has no study about the Mills type inequalities and Mills type ratios of skew-t distribution, and their applications to limiting distribution of extremes and large quantiles estimation. The aim of this short note is to fill this gap.

The rest of the paper is organized as follows. Mills type inequalities and Mills type ratios for skew-t distribution are derived in Section 2. Two applications are provided in Section 3 and Section 4 respectively. Section 3 establishes the asymptotic distributions of skew-t extremes. Methods and examples of large quantiles calculation of skew-t distribution are provided in Section 4.

2. Mills inequalities and Mills ratios

In this section, Mills inequalities and Mills ratios for $ST(\nu, \lambda)$ are derived. By using (2) and (3), and the symmetries of the pdf and the cdf of Student-t distribution, we have

$$-\frac{1}{x} \left(1 + \frac{x^2}{\nu}\right) \left(1 - \frac{\nu}{(\nu + 2)x^2}\right) < \frac{T_\nu(x)}{t_\nu(x)} < -\frac{1}{x} \left(1 + \frac{x^2}{\nu}\right) \tag{4}$$

for all $x < 0, \nu > 0$, and

$$\frac{T_\nu(x)}{t_\nu(x)} \sim -\frac{x}{\nu} \tag{5}$$

as $x \rightarrow -\infty$.

Theorem 1. Let $F_{\nu,\lambda}(x)$ and $f_{\nu,\lambda}(x)$ denote the cdf and the pdf of $ST(\nu, \lambda)$, respectively. For $x > 0$ and $\nu > 0$, we have

(i). for $\lambda > 0$,

$$\frac{1}{x} \left(1 + \frac{x^2}{\nu}\right) \left(1 + \frac{1}{x^2}\right)^{-1} < \frac{1 - F_{\nu,\lambda}(x)}{f_{\nu,\lambda}(x)} < \frac{1}{x} \left(1 + \frac{x^2}{\nu}\right) \left(1 - \frac{1}{\lambda x} \left(1 + \frac{\lambda^2 x^2}{\nu}\right) t_\nu(\lambda x)\right)^{-1}. \tag{6}$$

(ii). for $\lambda < 0$,

$$\begin{aligned} & \frac{x}{2\nu} \left(2 + \frac{2\nu}{x^2} - \left(1 - \frac{\nu}{(\nu + 2)\lambda^2 x^2}\right)^{-1} \left(1 + \frac{\nu}{x^2}\right)^{\frac{\nu+1}{2}} \left(1 + \frac{\nu}{\lambda^2 x^2}\right)^{\frac{\nu-1}{2}}\right) \left(1 + \frac{1}{x^2}\right)^{-1} \\ & < \frac{1 - F_{\nu,\lambda}(x)}{f_{\nu,\lambda}(x)} < \frac{x}{2\nu} \left(2 + \frac{2\nu}{x^2} - \left(1 + \frac{\nu}{x^2}\right) \left(1 + \frac{\nu}{\lambda^2 x^2}\right)^{-1}\right) \end{aligned} \tag{7}$$

for $\nu \geq 1$, and for $0 < \nu < 1$

$$\begin{aligned} & \frac{x}{2\nu} \left(2 + \frac{2\nu}{x^2} - \left(1 - \frac{\nu}{(\nu+2)\lambda^2 x^2} \right)^{-1} \left(1 + \frac{\nu}{x^2} \right) \left(1 + \frac{\nu}{\lambda^2 x^2} \right)^{\frac{\nu-1}{2}} \right) \left(1 + \frac{1}{x^2} \right)^{-1} \\ & < \frac{1 - F_{\nu,\lambda}(x)}{f_{\nu,\lambda}(x)} < \frac{x}{2\nu} \left(2 + \frac{2\nu}{x^2} - \left(1 + \frac{\nu}{\lambda^2 x^2} \right)^{-1} \left(1 + \frac{\nu}{x^2} \right)^{\frac{\nu+1}{2}} \right). \end{aligned} \tag{8}$$

Proof. For simplicity, let $C_\nu = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}}$. For $\nu > 0$ and $x > 0$, we have

$$\begin{aligned} 1 - F_{\nu,\lambda}(x) &= \int_x^{+\infty} 2t_\nu(y)T_\nu(\lambda y)dy \\ &= -\frac{2\nu C_\nu}{\nu+1} \int_x^{+\infty} \frac{1}{y} \left(1 + \frac{y^2}{\nu} \right) T_\nu(\lambda y) d\left(1 + \frac{y^2}{\nu} \right)^{-\frac{\nu+1}{2}} \\ &= \frac{\nu}{\nu+1} \frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) f_{\nu,\lambda}(x) - \frac{\nu}{\nu+1} \int_x^{+\infty} \frac{1}{y^2} f_{\nu,\lambda}(y) dy + \frac{1}{\nu+1} \int_x^{+\infty} f_{\nu,\lambda}(y) dy \\ &\quad + \frac{\nu}{\nu+1} \int_x^{+\infty} 2 \frac{\lambda}{y} \left(1 + \frac{y^2}{\nu} \right) t_\nu(y) t_\nu(\lambda y) dy, \end{aligned}$$

i.e.,

$$1 - F_{\nu,\lambda}(x) = \frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) f_{\nu,\lambda}(x) - \int_x^{+\infty} \frac{1}{y^2} f_{\nu,\lambda}(y) dy + \int_x^{+\infty} 2 \frac{\lambda}{y} \left(1 + \frac{y^2}{\nu} \right) t_\nu(y) t_\nu(\lambda y) dy. \tag{9}$$

In the case of $\lambda > 0$, by equation (9) we get

$$1 - F_{\nu,\lambda}(x) > \frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) f_{\nu,\lambda}(x) - \frac{1}{x^2} (1 - F_{\nu,\lambda}(x)), \tag{10}$$

and

$$\begin{aligned} 1 - F_{\nu,\lambda}(x) &< \frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) f_{\nu,\lambda}(x) + 2 \int_x^{+\infty} \frac{\lambda}{y} \left(1 + \frac{y^2}{\nu} \right) t_\nu(y) t_\nu(\lambda y) dy \\ &< \frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) f_{\nu,\lambda}(x) + \frac{2}{x} \left(1 + \frac{x^2}{\nu} \right) t_\nu(x) \int_x^{+\infty} \lambda t_\nu(\lambda y) dy \\ &= \frac{1}{x} \left(1 + \frac{x^2}{\nu} \right) f_{\nu,\lambda}(x) \frac{1}{T_\nu(\lambda x)}, \end{aligned} \tag{11}$$

where the second inequality holds as $(1 + y^2/\nu)t_\nu(y)1/y$ is decreasing function. Then (2), (10) and (11) implies (6) immediately.

In the case of $\lambda < 0$, we can obtain (7) and (8) similarly. Thus the proof of Theorem 1 is complete. \square

From Theorem 1 we can immediately get Mills ratio of $ST(\nu, \lambda)$ which states as follows.

Theorem 2. For $\nu > 0$, we have

(i).

$$\frac{1 - F_{\nu,\lambda}(x)}{f_{\nu,\lambda}(x)} \sim \frac{x}{\nu}, \text{ if } \lambda > 0; \quad \frac{1 - F_{\nu,\lambda}(x)}{f_{\nu,\lambda}(x)} \sim \frac{x}{2\nu}, \text{ if } \lambda < 0 \tag{12}$$

as $x \rightarrow \infty$;

(ii).

$$\frac{F_{\nu,\lambda}(x)}{f_{\nu,\lambda}(x)} \sim -\frac{x}{2\nu}, \text{ if } \lambda > 0; \quad \frac{F_{\nu,\lambda}(x)}{f_{\nu,\lambda}(x)} \sim -\frac{x}{\nu}, \text{ if } \lambda < 0 \tag{13}$$

as $x \rightarrow -\infty$.

Proof. The part (i) of Theorem 2 is an immediate result of Theorem 1. For the part (ii), it is easy to check the fact that

$$F_{\nu,-\lambda}(x) + F_{\nu,\lambda}(-x) = 1, \tag{14}$$

for any x . Then we can easily obtain (13) from (12). Therefore, the proof of Theorem 2 is complete. \square

Remark 1. By (3), (5) and Theorem 2, it is easily obtained that $F_{\nu,\lambda}(x)$ belongs to the max-domain of attraction of Fréchet distribution (written as $F_{\nu,\lambda} \in D(\Phi_\alpha)$ for the maxima and the minima-domain of attraction of Fréchet distribution (written as $F_{\nu,\lambda} \in D(1 - \Phi_\alpha(-x))$) for the minima.

Remark 2. Note that $1 - F_{\nu,\lambda}(x)$ is regular varying function, we can deduce from Theorem 3.29 of Foss et al. (2011) that $F_{\nu,\lambda}(x)$ is subexponential.

Remark 3. If we denote $q_{\nu,\lambda}(\theta)$ is the θ -th quantile of the skew-t distribution $ST(\nu, \lambda)$, then from (14) we have

$$q_{\nu,\lambda}(\theta) + q_{\nu,-\lambda}(1 - \theta) = 0. \tag{15}$$

3. Asymptotic distributions of extremes

In this section, we consider the limiting distributions of the partial maximum and the partial minimum of an independent and identically distributed sequence from skew-t distribution. Remark 1 shows that $F_{\nu,\lambda}(x) \in D(\Phi_\alpha(x))$ for maxima and $F_{\nu,\lambda}(x) \in D(1 - \Phi_\alpha(-x))$ for minima, where α depends on the sign of λ . Hence, finding the norming constants is the main work here. We have the following result.

Theorem 3. Let $X_n, n \geq 1$ be a sequence of independent skew-t random variables with common df $F_{\nu,\lambda}, \nu > 0$. Let $M_n = \max(X_k, 1 \leq k \leq n)$ and $m_n = \min(X_k, 1 \leq k \leq n)$ respectively denote the partial maximum and minimum. Then

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \leq a_n x) = \begin{cases} 0, & x \leq 0, \\ \exp(-x^{-\alpha}), & x > 0; \end{cases} \tag{16}$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P}(m_n \leq \alpha_n x) = \begin{cases} 1 - \exp(-(-x)^{-\beta}), & x < 0, \\ 1, & x \geq 0; \end{cases} \tag{17}$$

where norming constants a_n, α_n , and extreme value index α, β may depend on ν according to the sign of λ :

(i). for $\lambda > 0$,

$$a_n = \nu^{\frac{\nu-1}{2\nu}} (2C_\nu n)^{\frac{1}{\nu}} \quad \text{with} \quad \alpha = \nu$$

and

$$\alpha_n = \nu^{\frac{\nu-1}{2\nu}} \lambda^{-\frac{1}{2}} C_\nu^{\frac{1}{\nu}} n^{\frac{1}{2\nu}} \quad \text{with} \quad \beta = 2\nu;$$

(ii). for $\lambda = 0$,

$$a_n = \alpha_n = \nu^{\frac{\nu-1}{2\nu}} (C_\nu n)^{\frac{1}{\nu}} \quad \text{with} \quad \alpha = \beta = \nu;$$

(iii). for $\lambda < 0$,

$$a_n = \nu^{\frac{\nu-1}{2\nu}} (-\lambda)^{-\frac{1}{2}} C_\nu^{\frac{1}{\nu}} n^{\frac{1}{2\nu}} \quad \text{with} \quad \alpha = 2\nu$$

and

$$\alpha_n = \nu^{\frac{\nu-1}{2\nu}} (2C_\nu n)^{\frac{1}{\nu}} \quad \text{with} \quad \beta = \nu.$$

Proof. By arguments similar to Theorem 1.6.2 and Corollary 1.6.3 of Leadbetter et al.(1983), we can easily obtain the results of Theorem 3, thus we omit it here.

4. Large quantiles approximation

In this section, we will use the Mills inequalities and Mills ratios derived in Section 2 to approximate the large quantiles of skew-t distribution. In general, the $(1 - \theta)$ th quantile of skew-t distribution $ST(\nu, \lambda)$ is approximated by solving the following equation

$$1 - F_{\nu, \lambda}(x^*) = \theta, \tag{18}$$

for x^* . When θ is small, the resulted x^* is the approximated large quantile. Note that x^* is non-increasing in θ , i.e. $\theta \rightarrow 0$ implies $x^* \rightarrow \infty$. Therefore, by using the Mills' inequalities in Theorem 1, for the large quantile x^* , we can approximately calculate the large quantile x^* by solving the following equations

$$2 \left(\frac{1}{x^*} + \frac{x^*}{\nu} \right) t_\nu(x^*) T_\nu(\lambda x^*) = \theta, \quad \text{for } \lambda > 0, \tag{19}$$

and

$$2 \left(\frac{1}{x^*} + \frac{x^*}{2\nu} \right) t_\nu(x^*) T_\nu(\lambda x^*) = \theta, \quad \text{for } \lambda < 0, \tag{20}$$

where $t_\nu(x)$ and $T_\nu(x)$ is the pdf and cdf of standard student-t distribution with ν degrees of freedom, which can be readily computed by functions *dt* and *pt* in **R** software, respectively. To study the performances of the approximations for large quantiles with equations (19) and (20), we compute the $(1 - \theta)$ th quantiles of skew-t distributions $ST(\nu, \lambda)$ with $1 - \theta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ and 10^{-6} , $\nu = 3, 5$ and 10 , and $\lambda = 0.000001, 0.1, 1$ and 5 . The computed quantiles are summarized in Table 1. For comparisons, we also give the $(1 - \theta)$ th quantiles of standard Student-t distributions ($\lambda = 0$) and the left truncated Student-t distributions ($\lambda = \infty$), which has the pdf $2t_\nu(x) I_{(0, \infty)}(x)$ with $I(\cdot)$ being the indicator function. Therefore, the $(1 - \theta)$ th quantiles of the truncated Student-t distribution is exactly the $(1 - 2\theta)$ th quantiles of standard Student-t distributions, thus can be easily implemented by **R** software.

From Table 1, we can observe that as the shape parameter λ in the skew-t distribution $ST(\nu, \lambda)$ increases, the distribution is more right-skewed and the quantiles increase. And as the degree of freedom ν increases, the quantiles decreases. Moreover, when λ is very small, the quantiles of skew-t distribution $ST(\nu, \lambda)$ is very close to the associated quantiles the Student-t distribution ($\lambda = 0$) and as λ increases, the quantiles of skew-t distribution $ST(\nu, \lambda)$ is closer to the left truncated Student-t distribution ($\lambda = \infty$), which demonstrates the approximations by Mills' inequalities in equations (19) and (20) can give considerable accuracy of large quantiles of skew-t distributions.

ν	$1 - \theta$	λ					
		0	0.000001	0.1	1	5	∞
3	1e-2	4.540703	4.586887	5.119293	5.865954	5.876103	5.840909
	1e-3	10.214532	10.234343	12.227658	12.937385	12.939543	12.923979
	1e-4	22.203742	22.212892	27.677230	28.006816	28.007279	28.000130
	1e-5	47.927728	47.932452	60.307934	60.400035	60.400135	60.396824
	1e-6	103.299468	103.304019	130.134889	130.156128	130.156150	130.154590
5	1e-2	3.364930	3.415864	3.633673	4.067961	4.072794	4.032143
	1e-3	5.893430	5.919482	6.426086	6.890071	6.890806	6.868827
	1e-4	9.677566	9.692764	10.781279	11.190653	11.190768	11.177710
	1e-5	15.546855	15.556214	17.647778	17.904898	17.904916	17.896866
	1e-6	24.771030	24.776917	28.378731	28.483504	28.483507	28.478473
10	1e-2	2.763769	2.815116	2.929899	3.207941	3.210652	3.169273
	1e-3	4.143700	4.171223	4.348441	4.610349	4.610661	4.586894
	1e-4	5.693820	5.711512	5.977486	6.226818	6.226856	6.211051
	1e-5	7.526954	7.539396	7.930707	8.161607	8.161611	8.150287
	1e-6	9.751995	9.761198	10.324095	10.524920	10.524921	10.516490

Table 1: $(1 - \theta)th$ large quantiles of skew-t distribution $ST(\nu, \lambda)$ for $\lambda > 0$ approximated by Mills' inequalities.

References

Birnbaum, Z.M. (1942) "An inequality for Mills' ratio," *Annals of Mathematical Statistics*, 13, 245–246.

Boyd, A.V. (1959) "Inequalities for Mill's ratio," *Reports of Statistical Applications Research, Japanese Union of Scientists and Engineers*, 6, 44–46.

Foss, S., Korshunov, D. and Zachary, S. (2011) *An Introduction to heavy-tailed and subexponential Distributions*, Springer Verlag, New York.

Gafer, D.P. and Kafader, K. (1984) "A retrievable recipe for inverse t," *American Statistician*, 38(4), 308–311.

Leadbetter, M.R., Lindgren, G. and Rootzén, H. (1983) *Extremes and related properties of random sequences and processes*, Springer-Verlag, New York.

Liao, X., Peng, Z., Nadarajah, S. and Wang, X. (2012) "Rates of convergence of extremes from skew normal samples," Available on-line at <http://arxiv.org/abs/1212.1004>.

Mills, J.P. (1926) "Table of the ratio: area to bounding ordinate, for any portion of the normal curve," *Biometrika*, 18, 395–400.

Reiss, R.D. (1989) *Approximate distributions of order statistics: with applications to non-parametric statistics*, Springer, New York.

Sampford, M.R. (1953) "Some inequalities on Mill's ratio and related functions," *Annals of Mathematical Statistics*, 24, 130–132.

Schlüter, S. and Fischer, M. (2012) "A tail quantile approximation for the student t distribution," *Communications in Statistics - Theory and Methods*, 41, 2617–2625.

Soms, A.P. (1976) "An asymptotic expansion for the tail area of the t-distribution," *Journal of the American Statistical Association*, 71, 728–730.

Soms, A.P. (1980) "Rational bounds for the t-tail area," *Journal of the American Statistical Association*, 75, 438–440.