Performance Bounds for the Distribution-Generated Universal Portfolio

Choon Peng Tan
Department of Mathematical and Actuarial Sciences
Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia
E-mail: tancp@utar.edu.my

Abstract

Some well-known universal portfolios due to Cover and Ordentlich require implementation time and memory that are not practical in daily use. With this purpose in mind, the finite order universal portfolio is introduced requiring very much lesser time and memory for implementation due to its dependence only on a fixed finite number of past stock/asset data instead of storing the whole set of past data. In our context the earlier types of universal portfolios are referred to as moving-order universal portfolios. Some bounds for the ratio of wealths of the best constant-rebalanced portfolio (BCRP) to the universal portfolio are derived. The performance of the finite-order universal portfolio has been shown to be better than that of some previously well-known moving-order universal portfolios when it is run on some stock-price data sets. An algorithm for computing the distribution-generated universal portfolio is presented.

Keywords: investment returns, finite and moving order, BCRP, ratio of wealths

1. Introduction

Cover (1991) introduced the uniform universal portfolio and generalized this to the universal portfolio generated by the general class of probability distributions with support in the simplex of portfolio vectors in Cover and Ordentlich (1996) with special focus on the Dirichlet probability distribution. An extensive bibliography on the earlier work in universal portfolios is given in Cover (1991). The universal portfolio generated by the Dirichlet distribution at time $(n+1)$ has portfolio components which are normalized sums of all products of n different price relatives weighted by the $(n+1)^{th.}$ joint moments of the Dirichlet distribution. In our context, we refer to this portfolio as the moving-order Dirichlet universal portfolio and without loss of generality, we assume that the assets considered are stocks and the time (n+1) refers to the $(n+1)^{th.}$ trading day of the stocks. Ordentlich and Cover (1998) extended this concept to the universal portfolio generated by a sequence of probability mass functions forming a stochastic process in which the portfolio components at time $(n+1)$ are normalized sums of all products of n different price relatives weighted by the $(n+1)^{th.}$ joint probabilities of the generating distribution. An upper bound on the ratio of wealths of the best constant-rebalanced portfolio (BCRP) to the universal wealth behaving asymptotically as a polynomial in the number of trading days is derived in Ordentlich and Cover (1998). The moving-order universal portfolios mentioned so far are not practical in the sense that as the number of stocks in the portfolio increases, the implementation time and the computer storage requirements grow exponentially fast. A finite-order universal portfolio with comparable performance and requiring faster implementation time and much lesser computer memory is introduced in this paper.

2. Main Results

Consider a market of m stocks described by a sequence of price-relative vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n, \dots$ where the price-relative vector $\boldsymbol{x}_n = (x_{ni})$ on the $n^{th.}$ trading day consists of the $i^{th.}$ price relative x_{ni} which is the ratio of the closing price of the $i^{th.}$

stocks to its opening price on day n, for $i = 1,2,\dots,m$. We assume that $x_{ni} \geq 0$ for all $i = 1,2,\dots,m$ and all $n = 1,2,\dots$. Let $\boldsymbol{x}^n$ denote the sequence of n vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. The portfolio vector $\boldsymbol{b}_n = (b_{ni})$ on day n is the investment strategy used on day n where b_{ni} is the proportion of the current wealth invested on stock i for $i = 1,2,\dots,m$, with $0 \leq b_{ni} \leq 1$ and $\sum_{i=1}^{m} b_{ni} = 1$. Let $\hat{S}_n(\boldsymbol{x}^n)$ denote the universal wealth at the end of day n, given $\boldsymbol{x}^n$ and assuming that the universal portfolios $\widehat{\boldsymbol{b}}_1, \widehat{\boldsymbol{b}}_2, \dots, \widehat{\boldsymbol{b}}_n$ are used in this period. Furthermore, assuming that the initial wealth $S_o = 1$ unit, then

$$\hat{S}_n(\boldsymbol{x}^n) = \prod_{j=1}^{n} \widehat{\boldsymbol{b}}_j^t \boldsymbol{x}_j \ . \tag{1}$$

For a constant rebalanced portfolio, $\widehat{\boldsymbol{b}}_j = \boldsymbol{b}$ for all $j = 1,2,\dots$, and for some constant portfolio $\boldsymbol{b}$. The constant-rebalanced-portfolio wealth $S_n(\boldsymbol{x}^n)$ at the end of day n is

$$S_n(\boldsymbol{x}^n) = \prod_{j=1}^{n} \boldsymbol{b}_j^t \boldsymbol{x}_j \ . \tag{2}$$

Given $\boldsymbol{x}^n$, if (2) is maximized over all possible portfolio vectors $\boldsymbol{b} = (b_i)$ in the simplex $\mathcal{B}$ of portfolio vectors, then $S_n^*(\boldsymbol{x}^n)$ is known as the best constant-rebalanced-portfolio (BCRP) wealth, where

$$S_n^*(\boldsymbol{x}^n) = \max_{\boldsymbol{b}\in\boldsymbol{\mathcal{B}}} \left\{ \prod_{j=1}^{n} \boldsymbol{b}^t \boldsymbol{x}_j \right\} = \prod_{j=1}^{n} \boldsymbol{b}^{*t} \boldsymbol{x}_j \tag{3}$$

$$\mathcal{B} = \left\{ \boldsymbol{b}: 0 \leq b_i \leq 1, i = 1,2,\dots,m, \sum_{i=1}^{m} b_i = 1 \right\} \tag{4}$$

and $\boldsymbol{b}^*$in $\mathcal{B}$ achieves the maximum in (3).

Let $Y_1, Y_2, \dots, Y_m$ be m discrete/continuous random variables having a joint probability mass/density function $f(y_{1,\dots,}y_m)$ defined over the domain $\mathfrak{D}$ where

$$\mathfrak{D} = \{(y_1, \dots, y_m): f(y_1, \dots y_m) > 0\} \ . \tag{5}$$

Furthermore, let $\boldsymbol{y}$ denote the vector $(y_1, y_2, \dots, y_m)$. If $Y_1, Y_2, \dots, Y_m$ are mutually independent, we can have a mixture of discrete and continuous random variables. Given a positive integer ν, the *universal portfolio* $\widehat{\boldsymbol{b}}_{n+1}$ *of order* ν generated by $Y_1, Y_2, \dots, Y_m$ is defined as:

$$\hat{b}_{n+1,k} = \frac{\int_{\mathfrak{D}} y_k (\boldsymbol{y}^t \boldsymbol{x}_n)(\boldsymbol{y}^t \boldsymbol{x}_{n-1}) \dots (\boldsymbol{y}^t \boldsymbol{x}_{n-(\nu-1)}) f(y_1, \dots y_m) \ d\boldsymbol{y}}{\int_{\mathfrak{D}} (y_1 + \dots + y_m)(\boldsymbol{y}^t \boldsymbol{x}_n) \dots (\boldsymbol{y}^t \boldsymbol{x}_{n-(\nu-1)}) f(y_1, \dots y_m) \ d\boldsymbol{y}} \tag{6}$$

for $k = 1,2,\dots,m$ where we assume that $E\left[Y_1^{n_1} Y_2^{n_2} \dots Y_m^{n_m}\right] \geq 0$ for all non-negative integers $n_1, n_2, \dots, n_m$ satisfying $0 \leq n_i \leq \nu + 1$ for $i = 1,2,\dots,m$ and $\sum_{i=1}^{m} n_i = \nu + 1$. The numerator of $\hat{b}_{n+1,k}$ in (6) can be rewritten as:

$$\int_{\mathfrak{D}} y_k \left(\sum_{i_1=1}^{m} y_{i_1} x_{ni_1} \right) \left(\sum_{i_2=1}^{m} y_{i_2} x_{n-1,i_2} \right) \dots \left(\sum_{i_\nu=1}^{m} \left(y_{i_\nu} x_{n-\nu+1,i_\nu} \right) \right) f(y_1, \dots y_m) \ d\boldsymbol{y}$$

$$= \int_{\mathfrak{D}} y_k \left[\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \dots \sum_{i_\nu=1}^{m} \left(y_{i_1} y_{i_2} \dots y_{i_\nu} \right) \left(x_{ni_1} x_{n-1,i_2} \dots x_{n-\nu+1,i_\nu} \right) \right] f(y_1, \dots y_m) \ d\boldsymbol{y}$$

$$= \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \dots \sum_{i_\nu=1}^{m} \left(x_{ni_1} x_{n-1,i_2} \dots x_{n-\nu+1,i_\nu} \right) E\left[Y_1^{n_1(k;\boldsymbol{i})} Y_2^{n_2(k;\boldsymbol{i})} \dots Y_m^{n_m(k;\boldsymbol{i})} \right] \tag{7}$$

where $n_j(k;\boldsymbol{i})$ is the number of y_j's in the product $(y_k y_{i_1} y_{i_2} \dots y_{i_\nu})$, $\boldsymbol{i} = (i_1, i_2, \dots, i_m)$ for $1 \le i_j \le m$ for $j = 1,2,\dots,m$; $0 \le n_j(k;\boldsymbol{i}) \le \nu + 1$ and $\sum_{j=1}^{m} n_j(k;\boldsymbol{i}) = \nu + 1$. The denominator of $\hat{b}_{n+1,k}$ in (6) is the normalizing constant ζ_{n+1}^{-1}, where

$$\zeta_{n+1} = \left\{ \sum_{k=1}^{m} \left[\sum_{i_1=1}^{m} \dots \sum_{i_\nu=1}^{m} (x_{ni_1} \dots x_{n-\nu+1,i_\nu}) E\left[Y_1^{n_1(k;\boldsymbol{i})} \dots Y_m^{n_m(k;\boldsymbol{i})} \right] \right] \right\}^{-1}. \tag{8}$$

Thus

$$\hat{b}_{n+1,k} = \zeta_{n+1} \left\{ \sum_{i_1=1}^{m} \dots \sum_{i_\nu=1}^{m} (x_{ni_1} \dots x_{n-\nu+1,i_\nu}) E\left[Y_1^{n_1(k;\boldsymbol{i})} \dots Y_m^{n_m(k;\boldsymbol{i})} \right] \right\} \tag{9}$$

for $k = 1,2,\dots,m$. Note that ζ_{n+1} in (8) can also be written as:

$$\zeta_{n+1} = \left\{ \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \dots \sum_{i_\nu=1}^{m} (x_{ni_1} x_{n-1,i_2} \dots x_{n-\nu+1,i_\nu}) \times E\left[(Y_1 + Y_2 + \dots + Y_m) \left(Y_1^{n_1(\boldsymbol{i})} Y_2^{n_2(\boldsymbol{i})} \dots Y_m^{n_m(\boldsymbol{i})} \right) \right] \right\}^{-1} \tag{10}$$

where $n_j(\boldsymbol{i})$ is the number of y_j's in the sequence $y_{i_1}, y_{i_2} \dots y_{i_\nu}$ for $j = 1,2,\dots,m$; $0 \le n_j(\boldsymbol{i}) \le \nu$ and $\sum_{j=1}^{m} n_j(\boldsymbol{i}) = \nu$.

Lemma. *If $\xi_1, \xi_2, \dots, \xi_l$ and $\eta_1, \eta_2, \dots, \eta_l$ are $2l$ non-negative real numbers, then*

$$\min_j \left\{ \frac{\xi_j}{\eta_j} \right\} \le \frac{\sum_{i=1}^{l} \xi_i}{\sum_{i=1}^{l} \eta_i} \le \max_j \left\{ \frac{\xi_j}{\eta_j} \right\}.$$

Proof. See Cover and Ordentlich (1996) for the right inequality and Ordentlich and Cover (1998) for the left inequality.

Define the normalized joint moment of $Y_1^{n_1(k;\boldsymbol{i})}, \dots, Y_m^{n_m(k;\boldsymbol{i})}$ by $q_\nu(k;\boldsymbol{i})$ where

$$q_\nu(k;\boldsymbol{i}) = \frac{E\left[Y_1^{n_1(k;\boldsymbol{i})} \dots Y_m^{n_m(k;\boldsymbol{i})} \right]}{\sum_{l=1}^{m} E\left[Y_1^{n_1(l;\boldsymbol{i})} \dots Y_m^{n_m(l;\boldsymbol{i})} \right]}. \tag{11}$$

Then

$$q_\nu(k;\boldsymbol{i}) = \frac{E\left[Y_1^{n_1(k;\boldsymbol{i})} \dots Y_m^{n_m(k;\boldsymbol{i})} \right]}{E\left[(Y_1 + \dots + Y_m) Y_1^{n_1(\boldsymbol{i})} \dots Y_m^{n_m(\boldsymbol{i})} \right]}. \tag{12}$$

From (11),

$$q_\nu(k;\boldsymbol{i}) = \left\{ 1 + \sum_{\substack{l=1 \\ l \ne k}}^{m} \frac{E\left[Y_l^{n_l(\boldsymbol{i})+1} Y_k^{n_k(\boldsymbol{i})} \prod_{j \ne k,l} Y_j^{n_j(\boldsymbol{i})} \right]}{E\left[Y_k^{n_k(\boldsymbol{i})+1} Y_l^{n_l(\boldsymbol{i})} \prod_{j \ne k,l} Y_j^{n_j(\boldsymbol{i})} \right]} \right\}^{-1}. \tag{13}$$

If $Y_1, Y_2, \dots, Y_m$ are mutually independent,

$$q_v(k;\boldsymbol{i}) = \left\{1 + \sum_{\substack{l=1\\l\neq k}}^{m} \frac{E\left[Y_l^{n_l(\boldsymbol{i})+1}\right]E\left[Y_k^{n_k(\boldsymbol{i})}\right]}{E\left[Y_k^{n_k(\boldsymbol{i})+1}\right]E\left[Y_l^{n_l(\boldsymbol{i})}\right]}\right\}^{-1} = \frac{E\left[Y_k^{n_k(\boldsymbol{i})+1}\right]/E\left[Y_k^{n_k(\boldsymbol{i})}\right]}{\sum_{l=1}^{m} E\left[Y_l^{n_l(\boldsymbol{i})+1}\right]/E\left[Y_l^{n_l(\boldsymbol{i})}\right]}. \quad (14)$$

Assuming that the joint moments $E\left[Y_1^{n_1(k;\boldsymbol{i})} \ldots Y_m^{n_m(k;\boldsymbol{i})}\right]$ are strictly positive for all $0 \le n_j(k;\boldsymbol{i}) \le \nu + 1$ and $\sum_{j=1}^{m} n_j(k;\boldsymbol{i}) = \nu + 1$, we have $0 < q_v(k;\boldsymbol{i}) < 1$. Define

$$q_v = \min_{\substack{0 \le n_j(k;\boldsymbol{i}) \le \nu+1\\ \sum_{j=1}^{m} n_j(k;\boldsymbol{i}) = \nu+1}} \{q_v(k;\boldsymbol{i})\} \quad (15)$$

and hence $0 < q_v < 1$.

Proposition. *Consider the ratio of the BCRP wealth $S_n^*(\boldsymbol{x}^n)$ to the universal wealth $\hat{S}_n(\boldsymbol{x}^n)$ achieved by the universal portfolio of order ν generated by the m random variables $Y_1, \ldots, Y_m$ where the joint moments $E\left[Y_1^{n_1(k;\boldsymbol{i})} \ldots Y_m^{n_m(k;\boldsymbol{i})}\right] > 0$ for all $0 \le n_j(k;\boldsymbol{i}) \le \nu + 1$, $j = 1,2,\ldots,m$ and $\sum_{j=1}^{m} n_j(k;\boldsymbol{i}) = \nu + 1$. Then the ratio of wealths $\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)}$ satisfies the following inequality:*

$$\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)} \le q_\nu^{-n},$$

where q_ν is defined by (15).

Proof. From (9) and (10), the increase in wealth on day $(n+1)$ achieved by the universal portfolio is

$$\sum_{k=1}^{m} \hat{b}_{n+1,k} x_{n+1,k}$$

$$= \zeta_{n+1} \sum_{k=1}^{m} x_{n+1,k} \left\{ \sum_{i_1=1}^{m} \cdots \sum_{i_\nu=1}^{m} \left(x_{ni_1} \ldots x_{n-\nu+1,i_\nu}\right) E\left[Y_1^{n_1(k;\boldsymbol{i})} \ldots Y_m^{n_m(k;\boldsymbol{i})}\right] \right\}$$

$$= \sum_{k=1}^{m} x_{n+1,k} \frac{\sum_{i_1=1}^{m} \cdots \sum_{i_\nu=1}^{m} \left(x_{ni_1} \ldots x_{n-\nu+1,i_\nu}\right) E\left[Y_1^{n_1(k;\boldsymbol{i})} \ldots Y_m^{n_m(k;\boldsymbol{i})}\right]}{\sum_{i_1=1}^{m} \cdots \sum_{i_\nu=1}^{m} \left(x_{ni_1} \ldots x_{n-\nu+1,i_\nu}\right) E\left[(Y_1 + \cdots + Y_m) Y_1^{n_1(\boldsymbol{i})} \ldots Y_m^{n_m(\boldsymbol{i})}\right]}$$

By the Lemma,

$$\sum_{k=1}^{m} \hat{b}_{n+1,k} x_{n+1,k} \ge \sum_{k=1}^{m} x_{n+1,k} \min_{\substack{0 \le n_j(k;\boldsymbol{i}) \le \nu+1\\ \sum_{j=1}^{m} n_j(k;\boldsymbol{i}) = \nu+1}} \left\{ \frac{E\left[Y_1^{n_1(k;\boldsymbol{i})} \ldots Y_m^{n_m(k;\boldsymbol{i})}\right]}{E\left[(Y_1 + \cdots + Y_m) Y_1^{n_1(\boldsymbol{i})} \ldots Y_m^{n_m(\boldsymbol{i})}\right]} \right\}$$

$$= \sum_{k=1}^{m} x_{n+1,k}\, q_\nu \,. \quad (16)$$

Now the ratio of wealths

$$\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)} = \frac{\prod_{i=1}^{n} \boldsymbol{b}^{*t} \boldsymbol{x}_i}{\prod_{i=1}^{n} \hat{\boldsymbol{b}}_i^t \boldsymbol{x}_i} = \frac{\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_n=1}^{m} x_{ni_1} x_{n-1,i_2} \ldots x_{1,i_n}\; b_{i_1}^* b_{i_2}^* \ldots b_{i_n}^*}{\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_n=1}^{m} x_{ni_1} x_{n-1,i_2} \ldots x_{1,i_n}\; \hat{b}_{ni_1} \hat{b}_{n-1,i_2} \ldots \hat{b}_{1,i_n}}$$

$$\le \max_{i_1,\ldots,i_n} \left\{ \prod_{j=1}^{n} \frac{b_{i_j}^*}{q_\nu} \right\} \le q_\nu^{-n} \text{ by the Lemma and (16).}$$

Remarks. (i) A universal portfolio of order $\nu = n$ is known as a moving-order universal portfolio.

(ii) The bound given in the Proposition should be considered as a worst-case bound for the universal portfolio of order ν.

(iii) For the moving-order universal portfolio generated by the general Dirichlet $(\propto_1, \propto_2, \ldots, \propto_m)$ distribution, polynomial bounds in n for the ratio of wealths are derived in Tan (2002). These bounds are not asymptotic bounds for large n.

3. An Algorithm

In Tan (2004), an algorithm for computing the moving-order Dirichlet $(\propto_1, \propto_2, \ldots, \propto_m)$ universal portfolio is presented. In this section, we present an algorithm for computing a moving-order universal portfolio generated by m mutually independent random variables. This algorithm can be easily adapted to compute a finite-order universal portfolio where the order is large and for the case of dependent random variables.

Let $Y_1, Y_2, \ldots, Y_m$ be m mutually independent random variables generating a moving-order universal portfolio, where $\nu = n$. From (9) and (10), the portfolio $\{\hat{\boldsymbol{b}}_{n+1}\}$ is given by

$$\hat{b}_{n+1,k} = \zeta_{n+1}\left\{\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}\left(x_{1i_1}x_{2i_2}\ldots x_{ni_n}\right)E\left[Y_1^{n_1(k;\boldsymbol{i})}Y_2^{n_2(k;\boldsymbol{i})}\ldots Y_m^{n_m(k;\boldsymbol{i})}\right]\right\}, \qquad (17)$$

for $k = 1,2,\ldots,m$ and

$$\zeta_{n+1} = \left\{\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}\left(x_{1i_1}x_{2i_2}\ldots x_{ni_n}\right) \times E\left[(Y_1+Y_2+\cdots+Y_m)Y_1^{n_1(\boldsymbol{i})}Y_2^{n_2(\boldsymbol{i})}\ldots Y_m^{n_m(\boldsymbol{i})}\right]\right\}^{-1} \qquad (18)$$

where $n_j(\boldsymbol{i})$ is the number of y_j's in the sequence $y_{i_1}y_{i_2\ldots}y_{i_n}$ for $j = 1,2,\ldots,m;\ 0 \le n_j(\boldsymbol{i}) \le n$, $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$, $\sum_{j=1}^{m} n_j(\boldsymbol{i}) = n$; $n_j(k;\boldsymbol{i})$ is the number of y_j's in the sequence $y_k\ y_{i_1}\ y_{i_2}\ldots\ y_{i_n}$ for $j = 1,2,\ldots,m;\ 0 \le n_j(k;\boldsymbol{i}) \le n+1$, $\sum_{j=1}^{m} n_j(k;\boldsymbol{i}) = n+1$. We note that $n_j(k;\boldsymbol{i}) = n_k(\boldsymbol{i}) + 1$ if $j = k$ and $n_j(k;\boldsymbol{i}) = n_j(\boldsymbol{i})$ if $j \ne k$. The integer $n_j(\boldsymbol{i})$ is also the number of j's in the second subscript of x in the product $x_{1i_1}x_{2i_2\ldots}x_{ni_n}$. Let $\mathrm{X}_n(n_1, n_2, \ldots, n_m)$ denote the sum of all products $x_{1i_1}x_{2i_2\ldots}x_{ni_n}$ having the same set of counts $n_1(\boldsymbol{i}), n_2(\boldsymbol{i}), \ldots, n_m(\boldsymbol{i})$ summing up to n. In symbols,

$$\mathrm{X}_n(n_1, n_2, \ldots, n_m) = \sum_{n_1(\boldsymbol{i})+n_2(\boldsymbol{i})+\cdots+n_m(\boldsymbol{i})=n}\left(x_{1i_1}x_{2i_2}\ldots x_{ni_n}\right). \qquad (19)$$

Define $R_n(n_1, n_2, \ldots, n_m)$ as the following product of moments:

$$\mathcal{R}_n(n_1, n_2, \ldots, n_m) = \prod_{j=1}^{m} E\left[Y_j^{n_j}\right] \qquad (20)$$

where $n_1 + n_2 + \cdots + n_m = n$. Hence the numerator of $\hat{b}_{n+1,k}$ in (17) can be written as:

$$\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}\left(x_{1i_1}x_{2i_2}\ldots x_{ni_n}\right)E\left[Y_1^{n_1(k;\boldsymbol{i})}Y_2^{n_2(k;\boldsymbol{i})}\ldots Y_m^{n_m(k;\boldsymbol{i})}\right]$$
$$= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}\left(x_{1i_1}x_{2i_2}\ldots x_{ni_n}\right)E\left[Y_k^{n_k(\boldsymbol{i})+1}\right]\prod_{j\ne k}E\left[Y_j^{n_j(\boldsymbol{i})}\right]$$
$$= \sum_{n_1+n_2+\cdots+n_m=n}\mathrm{X}_n(n_1, n_2, \ldots, n_m)\mathcal{R}_{n+1}(n_1, \ldots, n_k+1, \ldots, n_m) \qquad (21)$$

where

$$\mathcal{R}_{n+1}(n_1, \ldots, n_k+1, \ldots, n_m) = E\left[Y_k^{n_k+1}\right]\prod_{j\ne k}E\left[Y_j^{n_j}\right]. \qquad (22)$$

The portfolio component $\hat{b}_{n+1,k}$ now becomes

$$\hat{b}_{n+1,k} = \frac{\sum_{n_1+n_2+\cdots+n_m=n} \mathrm{X}_n(n_1,n_2,\dots,n_m)\mathcal{R}_{n+1}(n_1,\dots,n_k+1,\dots,n_m)}{\sum_{j=1}^{m}\left[\sum_{n_1+n_2+\cdots+n_m=n} \mathrm{X}_n(n_1,n_2,\dots,n_m)\mathcal{R}_{n+1}(n_1,\dots,n_j+1,\dots,n_m)\right]} \quad (23)$$

for $k = 1,2,\dots,m$. The quantity $\mathrm{X}_n(n_1,n_2,\dots,n_m)$ is calculated recursively as follows:

$$\mathrm{X}_n(n_1,n_2,\dots,n_m) = \sum_{j=1}^{m} x_{nj}\, X_{n-1}(n_1,\dots,n_j-1,\dots,n_m) \quad (24)$$

with the initial conditions

$$\mathrm{X}_n(0,\dots,0,n,0,\dots,0) = x_{nj}\, X_{n-1}(0,\dots,0,n-1,0,\dots,0) \quad (25)$$

for $j = 1,2,\dots,m$. The moment function $\mathcal{R}_n(n_1,n_2,\dots,n_m)$ can be recursively calculated as:

$$\mathcal{R}_{n+1}(n_1,\dots,n_k+1,\dots,n_m) = \left(\frac{E\left[Y_k^{n_k+1}\right]}{E\left[Y_k^{n_k}\right]}\right) R_n(n_1,\dots,n_k,\dots,n_m) \quad (26)$$

for $n_k \geq 1, k = 1,2,\dots,m.$

The wealth function $\hat{S}_n(\boldsymbol{x}^n)$ can be calculated recursively as:

$$\hat{S}_{n+1}(\boldsymbol{x}^{n+1}) = (\hat{\boldsymbol{b}}_{n+1}\boldsymbol{x}_{n+1})\hat{S}_n(\boldsymbol{x}^n) \quad (27)$$

where from (6),

$$(\hat{\boldsymbol{b}}_{n+1}\boldsymbol{x}_{n+1}) = \frac{\int_{\mathfrak{D}} \left(\prod_{i=1}^{n+1} \boldsymbol{y}^t \boldsymbol{x}_i\right) f(y_1,\dots y_m)\ d\boldsymbol{y}}{\int_{\mathfrak{D}} (y_1+\cdots+y_m)\left(\prod_{i=1}^{n} \boldsymbol{y}^t \boldsymbol{x}_i\right) f(y_1,\dots y_m)\ d\boldsymbol{y}}$$

$$= \frac{\sum_{n_1+\cdots+n_m=n+1} \mathrm{X}_{n+1}(n_1,\dots,n_m)\mathcal{R}_{n+1}(n_1,\dots,n_m)}{\sum_{j=1}^{m}\left[\sum_{n_1+\cdots+n_m=n} \mathrm{X}_n(n_1,\dots,n_m)\mathcal{R}_{n+1}(n_1,\dots,n_j+1,\dots,n_m)\right]}. \quad (28)$$

The initial portfolio $\hat{\boldsymbol{b}}_1 = (b_{1k})$ can be chosen as $\hat{b}_{1k} = E(Y_k)/\sum_{j=1}^{m} E(Y_j)$ or $\hat{b}_{1k} = 1/m$ for $k = 1,2,\dots,m.$

4. Concluding Remarks

The performance of the Dirichlet $(\alpha_1,\alpha_2,\alpha_3)$ universal portfolis of order 1,2 and 3 has been studied by Tan, Chu and Lim (2012). It has been shown that the performance is comparable and in some cases better than that of the Cover-Ordentlich moving-order Dirichlet $(\alpha_1,\alpha_2,\alpha_3)$ universal portfolios in terms of the wealths achieved and faster implementation time.

References

[1] Cover, T.M. (1991). “Universal portfolios,” *Mathematical Finance* , 1, 1-29.

[2] Cover, T.M. and Ordentlich, E. (1996) “Universal portfolios with side informtion,” *IEEE Transactions on Information Theory*, 42, 348-363.

[3] Ordentlich, E. and Cover, T.M. (1998) “The cost of achieving the best portfolio in hindsight,” *Mathematics of Operations Research*, 23, 960-982.

[4] Tan, C.P. (2002) “Performance bounds for Dirichlet-weighted universal portfolios,” *Proceedings of the 2002 International Symposium on Information Theory and Its Applications*, Xi’an, China, 587-590, The Society of Information Theory and Its Applications, Tokyo.

[5] Tan, C.P. (2004) “Some developments in universal portfolios,” *Proceedings of the International Sri Lankan Statistical Conference*, Kandy, Sri Lanka, 495-505.

[6] Tan, C.P., Chu, S.Y. and Lim, W.X. (2012) “Dirichlet universal portfolios of finite order,” *The 16th. International Congress on Insurance: Mathematics and Economics*, Hong Kong, Book of Abstracts, 113.