Use of Çinlar Velocity Fields as a Subgrid Model

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Abstract

Large eddy simulation (LES) of the ocean flows is based on the numerical solution of larger eddies by filtered Navier-Stokes equations while modeling smaller ones. Çinlar velocity field which is inspired by eddies detected from Lagrangian studies of the ocean can efficiently represent the continuous change of eddy sizes from medium to smallest as a stochastic flow. The eddy parameters have been estimated from high-frequency radar observations verifying the model to be capable of generating mesoscale eddies up to 5-10km in radius. Therefore, it has been put forward as a subgrid model for LES. As a related model, the strained spiral vortex of Lundgren for turbulent fine structures in the ocean has already been demonstrated as a subgrid model. We use Çinlar velocity field composed of eddies of rotational form in two dimensions as it is verified to represent real data features. For the generalized version where the temporal decay depends on spatial variable, we have computed the energy spectrum using Gamma distribution for the eddy radius and shown that the data behavior in the wave number space is reflected as well. As for LES, typically the subgrid stress is modeled. From a statistical point of view, the covariance of the subgrid velocity with the resolved velocity in each step of LES can be estimated to represent the subgrid stress. Pursuing this idea, we have developed a numerical algorithm in OPENFOAM software taking into account the subgrid fluctuations modeled by our random velocity field. We report our first findings in this regard.

Key Words: Ocean modeling, subgrid scale, stochastic flow, random velocity field.

1. Introduction

Large eddy simulation (LES) corresponds to the numerical solution of filtered Navier-Stokes equations together with a model of smaller ones. The filtering operation is used to separate the large scales from small scales and the subgrid stress which remains unresolved is modeled by various approaches. In this paper, we consider Çinlar velocity field based on eddies from medium to smallest as a stochastic flow representing the subgrid scales.

Çinlar velocity field is composed of eddies which have random centers, radii and arrival times and decay exponentially in time to form a stationary, Markovian velocity field. The flow is incompressible and isotropic due to the form of the eddies. In [2], the decay rate is generalized to depend on the other parameters of an eddy. Let v be a deterministic velocity field on $\mathbf{R}^2$ called the basic eddy, and let $Q=\mathbf{R}^2\times\mathbf{R}\times(0,\infty)$ be an index set. Eddies of different sizes and amplitudes for $q\in Q,\ x\in\mathbf{R}^2$ are obtained by

$$v_q(x)=av\left(\frac{x-z}{b}\right),\ q=(z,a,b)$$

where q represents the type of an eddy and includes its center z in space, its amplitude a as well as its radius b. Let N be a Poisson random measure with mean measure

$$\mu(dt,dq)=\mu(dt,dz,da,db)=\lambda dtdz\alpha(da)\beta(db)$$

where λ is the arrival rate per unit time-unit space, and α and β are probability distributions. The arrival time *t* of an eddy, its center, amplitude and radius are all randomized with *N*. By the superposition of these eddies decaying exponentially in time, a stationary velocity field *u* is constructed as

$$u(x,t)=\int_{(-\infty,t)\times Q} N(ds,dz,da,db)e^{-c\left|\frac{x-z}{b}\right|^{2\gamma}(t-s)}av\left(\frac{x-z}{b}\right)$$

where $x\in\mathbf{R}^2, t\in\mathbf{R}, c>0, \gamma>0$, as the generalized form of Çinlar velocity field [2]. We consider an incompressible and isotropic flow in two dimensions. Therefore, the basic eddy $v=(v_1,v_2)$ $v=(v_1,v_2)$ is taken as a rotation around 0 with magnitude *m(r)* at distance *r* from 0, where $m:\mathbf{R}\to\mathbf{R}_+$ is continuous and has support [0,1]. The specific equations for *v* are

$$v_1(x)=-\frac{x_2}{r}m(r),\quad v_2(x)=\frac{x_1}{r}m(r)$$

where $x=(x_1,x_2)$ and $r=|x|\in[0,1]$.

The eddy parameters have been estimated from high-frequency radar observations verifying Çinlar model to be capable of generating mesoscale eddies up to 5-10km in radius [3]. Therefore, it has been put forward as a subgrid model for LES. In Section 2, we describe how Çinlar velocity field fits as a subgrid model. In Section 3, the algorithm for LES and a sample OPENFOAM run are given. Finally, the conclusions are stated in Section 4.

2. Subgrid Model

The turbulent flow is determined by Navier-Stokes equations for which exact solution is still impossible. The most precise approach for solution is direct numerical simulation, which requires representing all scales from the smallest to the largest. This is expensive in time and computer capacity. LES is an efficient approach based on numerical solution of larger eddies while only modeling the smaller ones. A filtering operation is used to separate the large scales from small scales (high frequency). Then, the subgrid stress which remains unresolved is modeled. Basic equations of fluid mechanics are obtained from the laws of conservation of mass and momentum. Resolved part $\overline{u(x,t)}$ of dependent variable $u(x,t)$ in these equations is defined by the following relationship:

$$\overline{u(x,t)}=\int_{-\infty}^{\infty} u(\xi,t')G(x-\xi,t-t')dt'd^3x$$

Navier-Stokes equations after the application of the filter in the following form:

$$\frac{\partial\bar{u}_i}{\partial t}+\frac{\partial\overline{u_iu_j}}{\partial x_j}=-\frac{\partial\bar{p}}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)$$

$$\frac{\partial\bar{u}_i}{\partial x_i}=0$$

The non-linear term $\overline{u_iu_j}$ should be written as a function of $\bar{u}$ and u'. The subgrid stress tensor $\tau_{ij}=\overline{u_iu_j}-\bar{u}_i\bar{u}_j$ is formed to get

$$\frac{\partial\bar{u}_i}{\partial t}+\frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial\bar{p}}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)-\frac{\partial\tau_{ij}}{\partial x_j} \tag{1}$$

Leonard decomposed this stress tensor as $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j = L_{ij} + C_{ij} + R_{ij}$ where

$$L_{ij} = \overline{\bar{u}_i \bar{u}_j}, \quad C_{ij} = \overline{\bar{u}_i u'_j} + \overline{\bar{u}_j u'_i}, \quad R_{ij} = \overline{u_i' u_j'}$$

and L_{ij}, the Leonard tensor, represents interactions among large scales, C_{ij}, represents cross-scale interactions between large and small scales and R_{ij}, the Reynolds stress-like term, represents interactions among the subgrid scales.

The strained spiral vortex of Lundgren [7] for turbulent eddies in the ocean has already been demonstrated as a subgrid model in [8]. Explicitly, the subgrid stress is given by $\tau_{ij} = K\left(\delta_{ij} - e_i^v e_j^v\right)$ where K is the turbulent kinetic energy in smaller scales which is estimated by using the energy spectrum of Lundgren vortices and the rate-of-strain tensor of the resolved flow. In this spirit, we have computed the energy spectrum of Çinlar velocity field using Gamma distribution for the eddy radius in [5]. The covariance tensor of the velocity field is found as

$$R_{ij}(x,t) = \frac{\lambda}{c} \int dz \int \alpha(da) a^2 \int \beta(db) \frac{b^2 e^{-c|z|^{2\gamma}|t|}}{|z|^{2\gamma} + |z + x/b|^{2\gamma}} v_i(z) v_j\left(z + \frac{x}{b}\right)$$

for $x \in \mathbf{R}^2$ and $t \in \mathbf{R}$. We choose the distribution β of b as a right-truncated Gamma distribution [5] given by

$$\beta(db) = \frac{b^{\theta-1} e^{-b/\zeta}}{\Gamma_{B/\zeta}(\theta)\zeta^{\theta}}, \quad 0 < b < B$$

where $\theta > 0$, $\zeta > 0$, and $\Gamma_{B/\zeta}(\theta)$ is the incomplete Gamma function with parameter θ and integration bounds from 0 to B/ζ, and only small scale eddies up to some cutoff B are considered. Turbulent energy per unit mass is given by

$$\frac{1}{2}\int_{\mathrm{R}^2} \sum_{j=1}^{2} \frac{1}{4\pi^2} \int e^{-ik.x} R_{jj}(x,0)\, dx\, dk =: \int_0^{\infty} \varepsilon(|k|)\, d|k|$$

where $\varepsilon(|k|)$ is the energy spectrum and a function of only $|k|$ due to isotropy. We have used the truncated Gamma distribution to find the form of the turbulent energy spectrum [5]. After some algebraic computations, we have shown that $\varepsilon(|k|)$ is proportional to

$$\varepsilon(|k|) \propto \frac{4\pi\lambda E(a^2)}{c\zeta^{\theta}\Gamma_{B/\zeta}(\theta)} |k|^{-\theta-4} F(B|k|)$$

for some function F, which is to be estimated from ocean data. The spectrum can be utilized as in [8] for estimating the subgrid stress τ_{ij}.

3. Subgrid Algorithm

The covariance of the subgrid velocity will be used to represent the subgrid stress in each step of LES as described in the previous section. The parameters of the radii and amplitude distributions together with the arrival and decay rates will be chosen to depend on the resolved velocity field. Then, the subgrid stress τ_{ij} can be written analytically as a function of the resolved velocity field $\bar{u}$. It follows that closure is obtained for the momentum equation (1) and one can solve it numerically for $\bar{u}$. Pisafoam solver in OPENFOAM, which is applicable to transient flows, can be used for this purpose.

The following numerical algorithm has been developed for Pisafoam in [6]:

1. Solve the discretized momentum equation implicitly for the predictor velocity using the current values of velocity and the pressure. For the first iteration the pressure from the previous time step is used.
2. Compute the mass fluxes based on the predictor velocity.
3. Assemble and solve the pressure-correction equation for the corrective pressure using the predictor velocity field.
4. Correct the mass fluxes to be divergence-free. This divergence-free field is used in the discretized convection term in the next time step.
5. Correct the predictor velocity explicitly to obtain a divergence-free velocity field.
6. Repeat steps 2-5 using the new velocity field as a predictor velocity for a set number of iterations.
7. Update boundary conditions, increase the time step and repeat from step 1.

A sample OPENFOAM output is given in Figure 1 for PitzDaily turbulence [4] using Pisafoam, based on the commonly used Smagorinsky subgrid model.

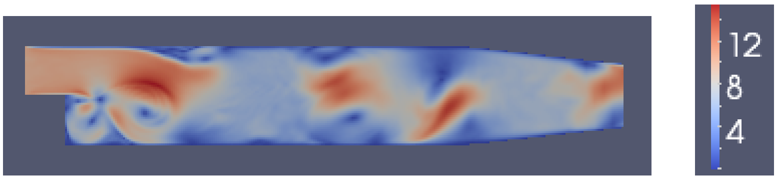

Figure 1. A snapshot of velocity magnitudes in OPENFOAM

4. Conclusions

A subgrid model for LES based on the covariance function of Çinlar velocity field has been proposed. Smagorinsky subgrid model in OPENFOAM has been experimented for a particular type of flow. Our proposal will be implemented similarly in homogeneous and isotropic flow as future work. It will be compared with commonly used subgrid models, and presence of backscatter, which is described as the energy flow from small to large scales, will be examined.

A natural extension of the proposed approach is accounting also for cross variances of the resolved velocity field with the subgrid velocity in the subgrid stress tensor [1]. Furthermore, kinetic energy will be investigated in space-time domain instead of the energy spectrum, since covariance structure is available in closed form for our velocity field. The results will be compared for numerical efficiency.

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