

Constraints and Restraints in Analysis and Design of Experiments with the Roles Defined Algebraically in Detail

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Abstracts

A prominence of General Linear Model (GLM) procedure of SAS/STAT (R) software so famous is the feasibility to be adapted to a variety of factorial arrangements, even to a crossed many-factorial one such that not only involving a cell or cells as of missing or plural runs, but also such that a cell or cells omitted or plurified, namely, adapted to the arrangement impaired or extended. Indeterminacy of solutions of the observation equation may be met with special solutions such as of Type I, II, III or IV as typified arbitrarily. Another possible way is such that the design matrix is to be augmented or condensed in the framework by constraints and/or restraints to secure a well-defined determinacy of the solutions as well as the algebraically linear orthogonality as to establish a unified and general scheme of analysis of response and variance.

Keywords: factorial design of experiments and the analysis, design matrix of general linear model (GLM) augmented or condensed by constraints and/or restraints

1. Introduction: algebraic procedure for statistical analysis

Actual data of real world are accompanied by fluctuations ever but a fact should be reminded that an algebraic ideal scheme itself is enormously complicated. It is so even if the actual data are truly free from any fluctuations just as shown by a response function $yy:AB...K(Aa, Bb, \dots, Kk)$ of the factors A, B, \dots, K . That is decomposed algebraically to the effect and contrast components, their elements, and other entities, by General Linear Model (GLM) procedure of the SAS/STAT (R) software or else.

1.1 General remarks on notations and expressions – a small grammar

1) A divisor ‘/’ means a product of all the items on the left is divided by the other product of all the items on the right just like in the common usage $V/ND = V/(ND)$.

2) A connector ‘@’ or ‘|’ leads suffice, sub- or super-scripts, exponents, remarks, or their trains, etc., whereas a terminator ‘#’ or ‘.’ ends that. By the rule, complication of an expression due to suffice, exponents, etc., often in small fonts, is avoided.

3) A Dirac’s ket (\rangle) marks the tail of a column vector, whereas a Dirac’s bra (\langle) is employed to mark the head of a row vector, either of linear function-vector algebra.

1.2 Function-vectors with the elements indexed mnemonically or symbolically

A response function $yy:X(Xx)$ of a factor X may be denoted in different ways such as $yy X(Xx)$ or so but almost equivalent and adapted or modified case to case.

a. $yy:X(Xx)\rangle$ is a ‘primitive’ column vector $:= yy:X(Xx) | x:=1, \dots, IX \rangle$ with each element indexed by the suffix x run from 1 to the number of levels IX of the factor X .

b. $yy:X(Xx)\rangle := yy:X(xi^{ix}) | ix:=1, \dots, IX \rangle$ with each element indexed by the mnemonic power xi^{ix} built of the mnemonic base xi . The exponent ix is set equal to the index x (the article **a.**) of each treatment levels of the factor X run from 1 to IX .

c. $yy:X((Xx)\rangle := yy:X((Xx) | x:=0, 1, \dots, IX \rangle$ with the primitive response column vector $yy:X(Xx)\rangle$ augmented by a null element $yy:X((Xx | x:=0)$ stacked above the top as it is enumerated by the index x set equal to 0. This convention is useful and the situation is marked sometimes by the left parenthesis doubled ‘((’ especially.

d. The power xi^{ix} (the article **b.**) may be replaced by another power xt^{tx} with the mnemonic base xi replaced by another symbolic base xt . The exponent tx is set

equal to the exponent ix minus 1 identically. It runs from -1 to the total number of degrees of freedom gX (of the factor X) that is defined equal to $IX-1$ identically.

- e. The rule applies to symbolic ‘contrast’ power XT^TX with the base XT , too.
- f. These conventions, used much flexibly or more freely case to case, are artificial and complicated but adapted well in practice and prove really convenient in analysis.

2. Truly observed response function-vector on a crossed n-factor complete layout

A run $AB...K(Aa,Bb, \dots, Kk)$ is settled by a set of levels Xx of n (in all) treatment factors X ($:= A,B, \dots, K$) with the suffix x ($:= 1, \dots, IX$ in all). It gives an observed response $yy:AB...K(Aa,Bb, \dots, Kk)$ for every run of the l ($:= lA.lB. \dots .lK$) runs listed in the dictionary order with the response yy building a true response column vector,

$$yy:AB...K(Aa, Bb, \dots, Kk) > := (yy:AB...K(A1, B1, \dots, K1) , \dots , yy:AB...K(A1A, B1B, \dots, K1K))' \quad (1)$$

Elements $yy:AB...K(Aa,Bb, \dots, Kk)$ are supposed to be a true variable dependent on a set of independent variables (Aa,Bb, \dots, Kk) and free from any true fluctuation vv .

2.1 Effect elements extracted from or defined of a response function

An ‘effect’ may be determined by 1) the additivity and 2) the isolability such that
 1) Additivity: $yy AB...K(Aa,Bb, \dots, Kk) = yy:M$ (general mean) $+yy:A(Aa) + yy:B(Bb) + \dots + yy:K(Kk)$ (main effects) $+yy:AB(Aa,Bb) + \dots + yy:HK(Hh,Kk)$ (two-factor int’ns) $+ \dots + yy:AB...K(Aa,Bb, \dots, Kk)$ (n-factor int’n: “yy:” cf. “yy” lefthand side), and (1a)
 2) Isolability: $\sum @a,b, \dots, k \# yy AB...K(Aa,Bb, \dots, Kk)^2 = ll. yy:M^2 + (1/lA). \sum @a \# yy:A(Aa)^2 + \dots + (1/lA.lB). \sum @a,b \# yy:AB(Aa,Bb)^2 + \dots + (1/lA.lB.lC). \sum @a,b,c \# yy:ABC(Aa,Bb,Cc)^2 + \dots + \dots + \sum @a,b, \dots, k \# yy:AB...K(Aa,Bb, \dots, Kk)^2$ provided $ll:=lA.lB. \dots .lK$, (1b)
 with the sums \sum of eqn (1b) applied on the crossed n-factor complete layout ever.

The additivity, eqn (1a), is not enough to determine uniquely effects, contrasts, the components, the elements, etc. The isolability, eqn (1b), should be supplemented.

2.2 Canonical or ‘usual’ constraints establishing the isolability and v.v.

Instead of the isolability, eqn (1b), the canonical or ‘usual’ constraints are often used to remove the indeterminacy of effects defined by the additivity, eqn (1a), only.

These constraints establish the isolability, eqn (1b), as well known, and, conversely, this is equivalent to them, as it is shown by careful steps of logical conception.

3. Algebraic analysis of experiment in 1-factor layout

A direct product of prior defined 1-factor functions $yy:X(Xx)$ (’s) can be a response function of a crossed n-factor complete layout. If the product is modulated precisely by some factors rr , that can represent any arbitrary n-factor function as follows,

$$yy:AB \dots K(Aa,Bb, \dots, Kk) := rr(Aa,Bb, \dots, Kk) @a \text{ modulation factor} \# . yy:A(Aa).yy:B(Bb). \dots .yy:K(Kk) \quad (1.1)$$

The 1-factor frameworks in sections 1.2, 3.1, 3.2, 3.3, 4.1, 4.2, 4.3 should be incorporated into any the n-factor frameworks in 1., 2.1, 2.2, 3. and 5.. consistently.

3.1 Augmented effect base matrix and the effect coefficient matrix

The response vector $yy:X(Xx) | x >$ (cf. section 1.2) of a factor X may be augmented by an element $yy:X((Xx) | @x=0\#)$ as postulated $\equiv 0$ and stacked above newly.

It forms an $(IX+1)$ -element column vector $yy:X((Xx) | @x:=0,1, \dots, IX\# >$ built of the effect component base vectors, $mcc:X[[jX]]((Xx) | x >, @jX=0, \dots, IX\#$, so that

$$yy:X((Xx) | x > = \sum @jX=0, \dots, IX\# mcc:X[[jX]]((Xx) | x >. cc:X[[jX]] \quad (2)$$

where the effect index (as named) jX is run from 0 to the total number of levels IX .

The topmost element $mcc:X[[jX]](X0)$ of each base vector $mcc:X[[jX]]((Xx) | x >$ is set $:= 0, 1, \dots, 1$ for $jX=0,1, \dots, IX$, one by one, respectively, to eliminate the topmost effect element $cc:X[[jX]=0]$ and to constrain the others $cc:X[[jX]@:=1, \dots, IX\#]$.

The row, $\langle jX | \text{mcc:X}[[jX] @ := 0, \dots, lX\#](X0)$, augments the matrix, mcc:X , to be square giving the effect coefficient matrix $\text{m}^*\text{cc:X}$ as the regular inverse, so that

$$\text{mcc:X} := [| x=0,1, \dots, lX \rangle \quad \text{mcc:X}[[jX](Xx) \langle jX=0,1, \dots, lX |] \quad \text{and} \quad (2a)$$

$$\text{m}^*\text{cc:X} := [| jX=0,1, \dots, lX \rangle \quad \text{m}^*\text{cc:X}[[jX](Xx) \langle x=0,1, \dots, lX |] \quad (2b)$$

mcc:X	$jX:=0$	1	2	...	lX	$\text{m}^*\text{cc:X}$	$x:=0$	1	2	...	lX
$x:=0$	0	1	1	...	1	$jX:=0$	-1/lX	1/lX	1/lX	...	1/lX
1	1	1	0	...	0	1	1/lX	1-1/lX	-1/lX	...	-1/lX
2	1	0	1	...	0	2	1/lX	-1/lX	1-1/lX	...	-1/lX
...
lX	1	0	0	...	1	lX	1/lX	-1/lX	-1/lX	...	1-1/lX

resulting in identities, with an $(lX+1)$ -dimensional unit matrix $(\sim lX \sim)$ defined here,

$$\text{mcc:X} . \text{m}^*\text{cc:X} = \text{m}^*\text{cc:X} . \text{mcc:X} = (\sim lX \sim) \quad (2c)$$

$$\text{therefore} \quad yy:X(Xx) | x \rangle = \text{mcc:X} . \text{cc:X}[[jX] | jX \rangle \quad (2c.1)$$

$$\text{and} \quad \text{cc:X}[[jX] | jX \rangle = \text{m}^*\text{cc:X} . yy:X(Xx) | x \rangle . \quad (2c.2)$$

The last equation establishes the effect elements $\text{cc:X}[[jX]$ of the 1-factor layout.

3.2 The contrast component base matrix and the contrast coefficient matrix

The base matrix $[| x \rangle \text{mcc:X} \langle jX |]$ and the conjugate $[| jX \rangle \text{m}^*\text{cc:X} \langle x |]$ may be replaced by the contrast component base matrix $[| x \rangle \text{mco:X} \langle TX |]$ and the contrast coefficient matrix $[| TX \rangle \text{m}^*\text{co:X} \langle x |]$, shown below as built of Irwin's vectors.

The response vector $yy:X(Xx) | @x:=0,1, \dots, lX\# \rangle$ is decomposed to a linear combination of the base vectors $\text{mco:X}[[XT^ATX](Xx) | x \rangle$ with the contrast exponent (as named) TX run from -1 to the number of degrees of freedom $gX(=lX-1)$.

$$yy:X(Xx) | @x:=0,1, \dots, lX\# \rangle = \sum @TX = -1,0, \dots, gX \# \text{mco:X}[[XT^ATX](Xx) | x \rangle @x:=0,1, \dots, gX\# . \text{co:X}[[XT^ATX] \quad (3)$$

$$\text{mco:X} := [| x=0,1, \dots, lX \rangle \text{mco:X}[[XT^ATX](Xx) \langle TX = -1,0, \dots, gX |] \quad (3a)$$

$$\text{m}^*\text{co:X} := [| TX = -1,0, \dots, gX \rangle \text{m}^*\text{co:X}[[XT^ATX](Xx) \langle x=0,1, \dots, lX |] \quad (3b)$$

mco:X	$TX:= -1$	0	1	2	...	$gX-1$	gX
$x:=0$	1	0	0	0	...	0	0
1	0	$1/\sqrt{lX}$	$1\%1$	$1\%2$...	$1/\sqrt{(3.2)}$	$1/\sqrt{(2.1)}$
2	0	$1/\sqrt{lX}$	$1\%1$	$1\%2$...	$1/\sqrt{(3.2)}$	$-1/\sqrt{(2.1)}$
3	0	$1/\sqrt{lX}$	$1\%1$	$1\%2$...	$-2/\sqrt{(3.2)}$	0
...
$lX-2$	0	$1/\sqrt{lX}$	$1\%1$	$1\%2$...	0	0
$lX-1$	0	$1/\sqrt{lX}$	$1\%1$	$-(lX-2)\%2$...	0	0
lX	0	$1/\sqrt{lX}$	$-(lX-1)\%1$	0	...	0	0

$\%1 := \sqrt{(lX.(lX-1))} \quad \%2 := \sqrt{((lX-1).(lX-2))} \quad gX := lX-1$

$\text{m}^*\text{co:X}$	$x:=0$	1	2	3	...	$lX-1$	lX
$TX:= -1$	1	0	0	0	...	0	0
0	0	$1/\sqrt{lX}$	$1/\sqrt{lX}$	$1/\sqrt{lX}$...	$1/\sqrt{lX}$	$1/\sqrt{lX}$
1	0	$1\%1$	$1\%1$	$1\%1$...	$1\%1$	$-(lX-1)\%1$
2	0	$1\%2$	$1\%2$	$1\%2$...	$-(lX-2)\%2$	0
...
$gX-1$	0	$1/\sqrt{(3.2)}$	$1/\sqrt{(3.2)}$	$-2/\sqrt{(3.2)}$...	0	0
gX	0	$1/\sqrt{(2.1)}$	$-1/\sqrt{(2.1)}$	0	...	0	0

$$\text{m}^*\text{co:X} := (\text{mco:X})^{(-1)} = (\text{mco:X})'$$

$$\text{mco:X} . \text{m}^*\text{co:X} = \text{m}^*\text{co:X} . \text{mco:X} = (\sim lX \sim) \quad \text{Cf. eqn (2c) ,}$$

resulting in identities $yy:X(Xx) | x \rangle = \text{mco:X} . \text{co:X}[[XT^ATX] | TX \rangle$

$$\text{and} \quad \text{co:X}[[XT^ATX] | TX \rangle (\$\$) = \text{m}^*\text{co:X} . yy:X(Xx) | x \rangle \quad (3c)$$

with the contrast elements $\text{co:X}[[XT^ATX]$ of the system defined here by $(\$\$)$.

3.3 Conversion matrix – Note. Some matrix is tagged newly as parenthesized.

The contrast component base matrix (GC:X) (:= mco:X_eqn(3a)) is got from the effect component base matrix (YY:X) (:= mcc:X_eqn(2a)) by a matrix (CCGC:X) (:= (CC:X)(:=m*cc:X_eqn(2b)). (GC:X)(:=mco:X_eqn(3a))) operated from the right.

The contrast coefficient matrix (CG:X) (:= m*co:X_eqn(3b)) is derived of the effect coefficient matrix (CC:X) (:= m*cc:X_eqn(2b)) by a matrix (CGYY:X) (:= (CG:X)(:=m*co:X_eqn(3b)).(YY:X)(:=mcc:X_eqn(2a))) operated from the left.

4 Use of the scheme applied to 1-factor layout

The descriptions so far in sections 3.1, 3.2 particularly and 3.3 have only outlined the scheme formally. The meaning and the use are detailed substantially below.

4.1 The design matrix extracted from the effect component base matrix

The matrix (YY:X), [|x:=0, ... , IX> mcc:X <jX:=0, ... , IX|], eqn(2a) and around, resumes to the primitive design matrix (JYY:X), as the topmost row removed, and in detail, [|x:=1, ... , IX> mcc:X[|jX|(Xx) <jX:=0, ... , IX|], IX rows×(IX+1) columns, to be of the observation equation of the effect elements, cc:X[|jX| | @jX:=0, ... , IX#,

$$yy:X(Xx) |x>_{primitive} = (JYY:X) . cc:X[|jX| | @jX:=0, ... , IX# >. \quad (4)$$

The column vector cc:X[|jX| | jX> is built of the topmost cc:X[|jX| | @jX:=0# for the average cc:XM, and of cc:X[|jX| | @jX:=1, ... , IX# below for the deviances.

The solution ia indeterminate and the constraint is called for. The response column vector yy:X(Xx) |x> is augmented by an artificial response element yy:X((X0). It is postulated identically 0 and stacked at the top to form a vector yy:X((Xx) |x>.

4.2 The orthonormal contrasts generated from the effect coefficient matrix

With the matrix (CC:X), [|jX:=0,1, ... ,IX> m*cc:X <x:=0,1, ... ,IX|], eqn (2b) and around, if the leftmost column removed, each row below the topmost depend linearly on the other (IX-1) rows whereas each IX rows are orthogonal to the topmost row.

The IX rows are turned by Irwin’s orthonormalization, one by one, finally to a set of linearly independent gX rows. The ‘number of degrees of freedom’ gX (:=IX-1) is discriminated from the ‘number of levels’ IX. A complete set of IX orthogonal rows in all is got, if the topmost row, isometric, added to the gX rows as the complement.

An IX-dimensional complete set of Irwin’s row vectors builds a contrast coefficient matrix (CG:X)_primitive, [|TX:= 0, ... ,gX> m*co:X <x:=1, ... ,IX|], eqn (3b) and around. It is relocated as (~CG:X~) by a unity ‘1’ for TX= -1 placed in the diagonal.

The matrix (CG:X) as operated on a response vector yy:(Xx) |x> generates the contrast element vector co:X(XT^TX) | @TX:=0, ... , gX #> _eqn (3b) and around.

4.3 The contrast base column vectors defined from the contrast coefficient matrix

The contrast component base matrix (GC:X) (:= (CG:X)’), mco:X, eqn (3a) and around, defines the vectors, mco:X[XT^TX | @TX:=0, ... , gX#](x) |x:=1, ... ,IX>.

The leftmost vector, mco:X[XT^0](x) |x:=1, ... ,IX>, isometric, is orthogonal to the others, mco:X[XT^TX |@TX:=1, ... ,gX#](x) |x:=1, ... ,IX>, orthogonal to each other: as renamed (GC:X[0] |x>) (average), (GC:X[1] |x>), ... ,(GC:X[gX] |x>) (deviances), and (<x| CG:X[0]), (<x| CG:X[1]), ... ,(<x| CG:X[gX]), the conjugates respectively.

A primitive response column vector is decomposed by them as follows:

$$yy:X(Xx) |x> = GC:X[0] |x>.co:X(XT^0) + \Gamma \quad (4a) \\ + GC:X[1] |x>.co:X(XT^1) + \dots + GC:X[gX] |x>.co:X(XT^gX)$$

as compared to another similar decomposition by the design matrix such that

$$yy:X(Xx) |x> = YY:X[0] |x>.cc:X(jX=0) + \Gamma \quad (4b) \\ + YY:X[1] |x>.cc:X(jX=1) + \dots + YY:X[IX] |x>.cc:X(jX=IX) .$$

The matrix (EX) (:= YY:X[0] |x> . <x| CC:X[jX=0]) is defined so that

$$(\$1) := (EX). yy:X(Xx) |x>, (\$2) := (DX). yy:X(Xx) |x>, (DX) := (IX)-(EX), \\ \text{and } (IX) := \sum @jX:=0,1, \dots ,IX# YY:X[jX] |x> . <x | CC:X[jX]) \quad \Gamma \quad (4c)$$

with an IX -dimensional unit matrix (\mathbf{IX}). The vectors (\$1) and (\$2) are such that
 (\$1) := $GC:X[0] |x\rangle.co:X(XT^0) = YY:X[0] |x\rangle.cc:X(jX=0)$. (4c.0)

$$GC:X[0] |x\rangle = YY:X[0] |x\rangle / \sqrt{IX} , \quad co:X(XT^0) = cc:X(jX=0) \cdot \sqrt{IX} .$$

$$cc:X(jX=0) = cc:XM = (1/IX) \cdot \sum_{@x=1, \dots, IX\#} yy:X(Xx) , \quad (4c.1)$$

$$($2) := GC:X[1] |x\rangle.co:X(XT^1) + \dots + GC:X[gX] |x\rangle.co:X(XT^gX)$$

$$= YY:X[1] |x\rangle.cc:X(jX=1) + \dots + YY:X[IX] |x\rangle.cc:X(jX=IX) , \quad (4c.2)$$

the contrast elements, $co:X(XT^TX) @TX:=0, \dots, gX\#$, remarked as a set of samples transformed of $yy:X(Xx) @x:=1, \dots, IX\#$ by the orthonormal matrix $(GC:X)'$: namely,

$$yy:X(Xx) |x\rangle = GC:X \cdot co:X(XT^TX) |TX:=0, \dots, gX\rangle , \quad \text{provided that } \Gamma \quad (4c.3)$$

$$GC:X = GC:X[0] |x\rangle + \dots + GC:X[gX] |x\rangle @ \text{“+.” for direct sum rightwards. } \# .$$

Sum of sq'd effect or contrast elements as equal to sum of squares of response

$$($3): \text{ defined here } := IX \cdot cc:X(jX=0)^2 + cc:X(jX=1)^2 + \dots + cc:X(jX=IX)^2 \quad (4c.4)$$

$$= co:X(XT^0)^2 + \dots + co:X(XT^gX)^2 = yy:X(X1)^2 + \dots + yy:X(XIX)^2 :$$

as it proves $IX \cdot cc:X(jX=0)^2 = co:X(XT^0)^2$, it shows every effect component base vector $YY:X[jX>0] |x\rangle_{eqn} (4c.2)$ is as if it were orthogonal to the others $\Gamma \quad (4c.5)$

$$($3.1): \quad co:X(XT^1)^2 + \dots + co:X(XT^gX)^2 = cc:X(jX=1)^2 + \dots + cc:X(jX=IX)^2 .$$

5. Conclusion: crossed n-factor layout, complete, impaired, abridged or extended

An n-factor layout is got as a direct product “ \times .” (or “ \cdot ”) of 1-factor layouts.

The n-factor response vector is defined here as follows (\$4)> :

$$($4)> := yy:AB \dots K:(a,b, \dots ,k) | a,b, \dots ,k \rangle \quad (\text{equal to a sum of the components})$$

$$:= \sum_{@jA,jB, \dots ,jK \#} yy:AB \dots K[jA, jB, \dots , jK](a, b, \dots , k) | a, b, \dots , k \rangle . \quad (5)$$

Each component vector $yy:AB \dots K[jA,jB, \dots ,jK](a,b, \dots ,k) | a,b, \dots ,k \rangle$ is equal to an effect component base vector (\$4.1) multiplied by an effect element (\$4.2).

$$($4.1) := YY:AB\dots K[jA,jB, \dots ,jK](a, b, \dots , k) | a,b, \dots ,k \rangle \quad (5.1)$$

$$:= YY:A[jA](a) | a \rangle \times YY:B[jB](b) | b \rangle \times \dots \times YY:K[jK](k) | k \rangle$$

$$($4.2) := cc:AB\dots K[jA,jB, \dots ,jK] | @ := \text{an element of the vector } ($4.3) \# . \quad (5.2)$$

The n-factor effect element column vector (\$4.3) is defined as follows

$$($4.3) := cc:AB\dots K[jA,jB, \dots ,jK] | jA,jB, \dots ,jK \rangle := (CC:AB\dots K) \cdot ($4)> \quad (5.3)$$

$$= (CC:AB\dots K) \cdot yy:AB\dots K:(a,b, \dots ,k) | a,b, \dots ,k \rangle .$$

where the n-factor effect coefficient matrix is defined (\$4.4) to be

$$($4.4) := (CC:AB\dots K) := \sum_{@jA, jB, \dots , jK \#} \langle a | CC:A[jA](a) \cdot \times$$

$$\langle b | CC:B[jB](b) \cdot \times \dots \cdot \times \langle k | CC:K[jK](k) \cdot \times$$

$$= (CC:A) \cdot \times (CC:B) \cdot \times \dots \cdot \times (CC:K) \quad (5.4)$$

whereas the n-factor effect component base matrix is defined (\$4.5) to be

$$($4.5) := (YY:AB\dots K) := \sum_{@jA, jB, \dots , jK \#} \text{ the base vector } ($4.1) .$$

$$= (YY:A) \times (YY:B) \times \dots \times (YY:K) \quad (5.5)$$

(\$4.6) : Either “ \sum ” or “ Σ .” for downwards or rightwards direct sum, respectively.

$$CC:X := \sum_{@jX \#} \langle x | CC:X[jX](x) , \quad YY:X := \sum_{@jX \#} YY:X[jX](x) | x \rangle \quad (5.6)$$

A quadratic form (\$5) of the vector $yy:AB \dots K:(a,b, \dots ,k) | a,b, \dots ,k \rangle$ is to be

$$($5) := \langle a,b, \dots ,k | yy:AB \dots K:(a,b, \dots ,k) \cdot yy:AB \dots K:(a,b, \dots ,k) | a,b, \dots ,k \rangle$$

$$= \langle jA,jB, \dots ,jK | cc:AB\dots K[jA,jB, \dots ,jK] \cdot ((YYYY:AB\dots K) @ := ($6) \#)$$

$$\cdot cc:AB\dots K[jA,jB, \dots ,jK] | jA,jB, \dots ,jK \rangle \quad \text{provided that} \quad (5a)$$

$$($6) := (YYYY:AB\dots K) := (YY:AB\dots K)' \cdot (YY:AB\dots K) = (YY:AB\dots K)^2 \quad \text{and}$$

$$(YYYY:AB\dots K) = (YYYY:A) \times (YYYY:B) \times \dots \times (YYYY:K) \quad (5b)$$

The 1-factor quadratic form matrix (YYYY:X) (\$6) is decomposed as follows:

$$($7):= (YYYY:X) := (YY:X)^2 = (YYEE:X) + (YYDD:X) + (YYNN:X) \quad (5c)$$

where (YYEE:X) := (0, 1, 1, ..., 1)' + (0, 1, 1, ..., 1) ,

(YYDD:X) := diagonal matrix [0, 1, 1, ..., 1] , and

(YYNN:X) := (0, 1, 1, ..., 1)' \cdot (0, 1, 1, ..., 1) ; each vector interpreted

to be an (IX+1)-dimensional square matrix all supplemented with null “0” elements..

5.1 Quadratic form of an n-factor crossed complete layout

The matrix (YYYY:AB...K), (56), is a direct product of 1-factor matrixes, eqn (5b), that produces the quadratic form, eqn (5a), with multipliers on both side. This leads the equation of isolability, eqn (1b). Its terms are got of the 1-factor matrix (YYYY:X), and the parts (YYEE:X), (YYDD:X), (YYNN:X), eqn (5c). Some terms vanish as to be confirmed concretely and exactly in frameworks of direct products and sums.

5.2 Impaired layout caused from omission of treatment, observation or cell

Under a circumstance, the observed system, in nature, shall omit some treatment of the n-factor crossed complete layout, and therefore the observation or the cell.

The observation equation of the cell may play like an extraordinary constraint with the identically zero response. The effect component of the highest order is equated to a sum of the lower orders. The observation equation of the n-factor layout as augmented by all the constraints is rebuilt consistently with the '1' degree of freedom lost.

This impaired layout is processed similarly as by SAS STAT(R) PROC GLM but the solution is to be definite uniquely as that in an n-factor crossed complete layout.

5.3 General Linear Model (GLM) procedure with the solutions indefinite

Without ordinary i.e. 'usual' or canonical constraints, the solution is to be indefinite and some tools are required just like the sums of squares Type I, II, III, IV or others.

Detailed normally in SAS GLM User's Guide etc., the most important rule of every pivot selection in forward and backward calculations is unclear, still, perhaps as it is not interesting to anybody but it looks cumbersome and burdensome, if described.

The rule, however, if opened explicitly, should help the appropriate comprehension of each sums of squares, the effect components and the coefficients very correctly.

5.4 Extended layout by plural inclusion of treatments, observations or cells

The SAS STAT(R) PROC GLM, shown in SAS Learning Edition 4.1 (sold and was alive formerly), provides some sequential decompositions (Type I) for the extension.

The scheme differently proposed here is suitable, too, to the extensions, if Irwin's orthonormalization is applied to a necessary part of an extended response vector.

5.5 Fisher's z or F test in terms of sums of squared contrast elements

A set of contrast elements is orthonormally transformed of a set of responses as it is shown in eqn (4c.3). The z test or F test is constructed of the ratios of χ^2 ('s) of the transformed set regarded to be a set of independent samples of a normal distribution.

5.6 Functional response representing stochastic response, samples and population

An algebraic scheme is built without reference to any statistical entity. A sample is regarded and treated like a functional entity. Still it is much complicated as outlined.

A stochastic entity is required if to be extrapolated to any 'future' in a population so stable to be predictable. The scheme should differ from an algebraic scheme.

A really observed response y is postulated to be a sum of a true response yy and a true fluctuation vv. Cast on an n-factor complete layout, an effect element of yy is often identically 0, then, the element of y contains only vv and this be measurable.

5.7 Restraints resulting varieties of abridged n-factor layouts and the uses

A crossed n-factor complete layout is downsized remarkably by restraining some interaction effect components as postulated axiomatically to be identically zero. It is useful in exploration, observation and operation if proved in particular treatments.

5.8 Non-isometric main base vector and the constraint in a 1- or n-factor layout

Such a non-isometric may be called "polycline" (, say), as compared to "canonical", defining a weighted mean and the constraint but with some complication, of course.

References

Irwin, J.O. (1934): On the independence of the constituent items in the analysis of variance., J.R.S.S. Industrial and Agricultural Research Supplement., 1, 936-951.