The power of tail independence tests in extreme value models. An application for stock exchange markets

Grażyna Trzpiot[1], Justyna Majewska[23]
[1] University of Economics in Katowice, POLAND
[2] University of Economics in Katowice, POLAND
[3] Corresponding author: Justyna Majewska, e-mail: justyna.majewska@ue.katowice.pl

Abstract

Dependencies of extreme events are attracting an increasing attention in modern risk management. In practice, the concept of tail dependence represents the current standard to describe the amount of extremal dependence. This paper presents some of the important issue of testing for tail independence. Applied tests are based on Extreme Value Theory. While Extreme Value Theory allows to construct estimators of the tail dependence coefficient and to derive tests for tail independence, the determination of the threshold where the tails begin remains a fundamental statistical problem. One important feature of this paper is carrying out an simulation study (according to percentage of extreme observations) to compare results of analyzed tests. Given the results of the simulation, the application part is concerned on time series from selected worldwide stock exchange markets.

Keywords: extreme value theory, extreme dependence, tail dependence coefficient

1. Introduction

Estimating dependence between risky asset returns is the cornerstone of portfolio theory and many other finance applications. Common dependence measures such as Pearson's correlation coefficient are not always suited for a proper understanding of dependencies in financial markets, Embrechts et al. (2002). In particular, dependencies between extreme events such as extreme negative stock returns or large portfolio losses cause the need for alternative dependence measures to support asset-allocation strategies. Several empirical surveys such as Ane, Kharoubi (2003) and Malevergne, Sornette (2004) exhibited that the concept of *tail dependence* is a useful tool to describe the dependence between extremal data. Tail dependence is described via the *tail-dependence coefficient* introduced by Sibuya (1960). Extreme value theory is the natural choice for inferences on extreme values. In this paper, we are concerned with *testing* for pairwise independence of maxima from empirical data, which seem to be absolutely mandatory for tail dependence *estimation*. The aim of the paper is presentation of tests for tail independence, which is indispensable when working with tail dependence, since all estimators of the tail dependence coefficient are strongly misleading when the data does not stem from a tail dependent setting.

1. Tail dependence concept

The *tail dependence coefficient* is roughly speaking the probability that a random variable exceeds a certain threshold given that another random variable has already exceeded that threshold. The following approach, Sibuya (1960) and Joe (1997) among others, represents the most common definition of tail dependence. Let (X,Y) be a random pair with joint cumulative distribution function F and marginals F_x and F_y. The quantity $\lambda_u = \lim_{v \to 1-} P(X > F_X^{-1}(v) \mid Y > F_Y^{-1}(v))$ is the *upper tail-dependence* *coefficient* (upper TDC), provided the limit exists. We say that (X,Y) is upper tail dependent if $\lambda_u > 0$ and upper tail independent if $\lambda_u < 0$. Similarly, we define the lower tail-dependence coefficient λ_L.

The TDC can also be defined via the notion of copula, introduced by Sklar (1959). A copula C is a cumulative distribution function whose margins are uniformly distributed on [0, 1]. The joint distribution function F of any random pair (X,Y) can be represented as $F(x) = C(F_X(x), F_Y(y))$ (refer to Joe (1997) for more information on copulas). The coefficient of upper tail dependence can be written in terms of copula:

$$\lambda_U = \lim_{v \to 1-} \frac{1-2v+C(v,v)}{1-v}. \qquad (1)$$

Analogously, we have $\lambda_L = \lim_{v \to o+} \dfrac{C(v,v)}{v}$.

2. Bivariate extreme distributions

The classical extreme bivariate theory is concerned with the limit behaviour of $(M_n(X), M_n(Y)) = (\max_{i=1,\dots n} X_i, \max_{i=1,\dots n} Y_i)$ as $n \to \infty$. Because of the definition, the marginals of $(M_n(X), M_n(Y))$ belong to the generalized extreme value (GEV) distribution family. The general form of a generalized extreme value GEV distribution is $GEV_{\mu,\sigma,\xi}(x) = \exp(-[1+\xi \frac{x-\mu}{\sigma}]^{-1/\xi})$ with $\mu \in R$, $\sigma > 0$, $\xi \in R$ (Coles 2001). To simplify the presentation, Coles (2001) assumes without loss of generality that $F_X \equiv F_Y \equiv F$, where $F(\cdot)$ is the unit Frechet distribution. The following theorem (de Haan and Resnick, 1977) characterizes the limit joint distribution of $(M_n(X), M_n(Y))$:

If $P(M_n(X) \le nx, M_n(Y) \le ny) \xrightarrow[n \to \infty]{} G(x,y)$ where G is a non-degenerate distribution function, then $G(\cdot,\cdot)$ takes the form $G(x,y) = \exp(-V(x,y))$ with $V(x,y) = 2\int_0^1 \max(\omega/x, (1-\omega)/y) dH(\omega)$ and H is a distribution on $[0,1]$ with mean $1/2$.

3. Estimation of the TDC

There are two possibilities to use Extreme Value Theory for the estimation of the TDC. The first one is to develop estimators based on the assumptions of the Generalized Pareto Distribution. Therefore, one assumes convergence (over some threshold) to a bivariate Generalized Pareto Distribution. This model is called Peaks over Threshold. The other possibility is to assume that the assumptions of the GEV are fulfilled. This conception in a financial application rarely be the case. Both methods come to the same estimation problem: the dependence function is to be estimated. The difference is the treatment of the data: in the first case, we choose the realizations that lie above a threshold, in the second case - block-maxima. Frahm et al. (2005) give estimators for the TDC under different assumptions: using a specific distribution (e.g. t-distribution), within a class of distributions (e.g. elliptically contoured distributions), using a specific copula (e.g. Gumbel), within a class of copulae (e.g. Archimedean) or a nonparametric estimation (without any parametric assumption). The authors compare the performance of the different estimators for different cases: whether the assumption is true or wrong and whether there is tail dependence or not. It turns out that some of the estimators perform well if there is tail dependence but bad if there is not. In practical applications, one will never know which copula model is the correct one. The estimation can only be under misspecification. So difficulties in selecting a copula model, brings us to the important issue of testing for tail dependence.

4. A different approaches for testing for tail independence

One of the most interesting approach for testing for tail independence is given in Falk and Michel (2006). They prove the following theorem:

With $c \to 0$, we have uniformly for $t \in [0,1]$:

$$P(X+Y > ct \mid X+Y > c) = \begin{cases} t^2; & \textit{there is no tail dependence} \\ t; & \textit{else} \end{cases}.$$

Using this theorem, Falk and Michel propose four different tests for tail independence, which can be grouped into 2 different classes: a Neymann-Pearson test (NP) and three goodness of fit tests: Fisher's κ, Kolmogorov-Smirnov and χ^2. In the latter class, the Komolgorov-Smirnov-test (KS) turns out to be the best in the simulation study by Falk and Michel (2006). Therefore, in the following, only NP and KS tests are described.

Neyman-Pearson test

Assume we have a random sample $(X_1,..,X_n)$ $(Y_1,..,Y_n)$ of independent copies of (X,Y). The marginal distribution is assumed to be reverse exponential (i.e. $F(x,0) = F(0,x) = \exp(x)$). Now, fix a threshold $c < 0$ and consider $E = \{C_i = X_i + Y_i; C_i > c\}$. Let $K(n) = \# E$ and define $V_i = C_i / c \ \ \forall i = 1,...,k(n)$. The NP test considers the distribution function of V_i and tests whether it is more likely from $F_{(0)}(t) = t^2$ or $F_{(1)}(t) = t$. The test statistic for testing $F_{(0)}$ (tail independence) against $F_{(1)}$ is (for fixed n):

$$T_{NP} := \log\left(\prod_{i=1}^{k(n)} \frac{1}{2V_i}\right) = -\sum_{i=1}^{k(n)} \log(V_i) - k(n)\log(2).$$

$F_{(0)}$ is rejected when T_{NP} gets large precisely, if the approximate p-value $p_{NP} := \Phi(k(n)^{-1/2} \sum_{i=1}^{k(n)} (2\log(V_i)+1))$ is too close to 0, typically if $p_{NP} \le .05$; Φ - standard normal df.

Kolmogorov Smirnov test

A different possibility of using Falk and Michel (2006) theorem is to carry out a goodness-of- fit test, in this case using the Kolmogorov Smirnov test. Therefore, define, conditional on $K(n) = m$:

$U_i = F_c(C_i / c) = (1-(1-C_i)\exp(C_i))/(1-(1-c)\exp c)$, $\forall i \in \{1,...m\}$. Denote $\hat{F}_m(t) = \frac{1}{m} \sum I_{[0,t]} C_i$ the ecdf of U_i, $i = 1,..,m$. The Kolmogorov test statistic is then:

$$T_{KS} := \frac{1}{m} \sup_{t \in [0,1]} \left| \hat{F}_m(t) - t \right|.$$

The approximate p-value is $p_{KS} = 1 - K(T_{KS})$, where K is the cdf of the Kolmogorov distribution. According to a rule of thumb given by the authors: for $m > 30$, tail independence is rejected if $T_{KS} > c_{0,05} = 1{,}36$.

Another approach for testing for tail independence is given in Draisma et al. (2004) and this is a **test based on the residual dependence index**. They present three different estimators for the residual dependence index η : a maximum likelihood estimator in a Generalized Pareto model, a Hill estimator and the estimator presented in Peng (1999). Here, Peng's estimator turns out to be outperformed by the other two estimators in the simulation study by Draisma et al. (2004). The Hill estimator is presented since it can be easily implemented and as Draisma et al. (2004) argue, the maximum likelihood's advantage of location invariance over the Hill estimator is not relevant. Furthermore the Hill estimator has lower variance. The Hill estimator is defined as:

$$\hat{\eta} = \frac{1}{n}\sum_{i=1}^{m} \log \frac{T_{n,n-1+i}^{(n)}}{T_{n,n-m}^{(n)}}$$

where $T_{n,k}^{(n)}$ is the k-th order statistic of

$$T_i^{(n)} = \min\left(\frac{n+1}{n+1-R_i^X};\frac{n+1}{n+1-R_i^Y}\right).$$

Figure 1 presents the estimation results for $\hat{\eta}$ for Gaussian distributions with different parameters. The full line always represents the case of tail dependence ($\hat{\eta}$ = 1), the dotted line corresponds to the mean of the estimation over 1000 simulations, the dashdotted lines are the confidence intervals bounds (at 10%). Therefore, in a onesided test, the null hypothesis of tail-independence can be rejected if the upper dash-dotted line is above the full line (i.e. 1). So for the Gaussian distribution with ρ = 0.8 we can reject tail independence but for lower correlation we are able to accept it.

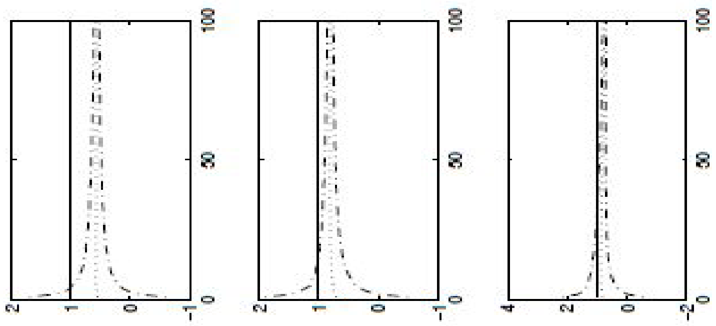

Figure 1. Estimation results of $\hat{\eta}$ (dotted) and confidence intervals (10%, dash-dotted) on the *x*-axis and *m* on the *y*-axis for Gaussian (ρ= 0.2, ρ = 0.5 and ρ = 0.8, from left to the right).
Source: Till Großmaß , Copulae and tail dependence, Diploma thesis, Berlin, 2007

The figure 1. illustrates the variance-bias trade-off: the higher the threshold, the lower the variance but the higher the bias.

5. Simulation results

In this section the power of the extreme-value dependence tests are examined. In order to examine this issue we carry out Monte Carlo experiments. Each of the experiments consist of generating two returns series of 1000 observations each from GARCH(1, 1) processes, whose joint behavior is assumed to be adequately represented by a SJC copula and *t* copula with 5 degrees of freedom. Each of the Monte Carlo experiments is repeated 100 hundred times and tests NP, KS and test based on the residual dependence index (RDI) are computed for the lower tails at each iteration. Our results are reported in Table 1.

Table 1. Simulation of failure to reject
H_0: tail independence (TYPE II ERROR)

% OF EXTREME OBSERVATIONS IN TAIL	NP TEST	KS TEST	RDI TEST
1	1%	1%	3%
2	3%	3%	5%
…	…	…	…
11	4%	6%	12%
12	5%	7%	15%

Simulations indicate that the Neyman–Pearson test (NP) has the smallest type II error rate, closely followed by the Kolmogorov–Smirnov. The RDI test does not control the type II error rate if the percentage of extreme observations is high.

6. Empirical results

Given the results of the simulation, we can now turn to the empirical analysis. The way the data is transformed via a GARCH(1,1) process. Four different data sets are analyzed, which cover different sectors: shares of high technology enterprises, automobile producers, currencies and stock market indices.
The analysis are carried out using day-today log-returns and a sliding window with window length of 250 data points.
For the purpose of estimation we used empirical copula. Empirical counterpart of λ_l can be obtained by plugging the empirical copula into Eq. (1):

$$\hat{C}(\frac{i}{n},\frac{j}{n}) = \frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{\{u_k \le u(i), v_k \le v(j)\}} \; ; \; i, j = 1,2,...,n$$

$u_{(1)} \le u_{(2)} \le ... \le u_{(n)}$ and $v_{(1)} \le v_{(2)} \le ... \le v_{(n)}$ are the order statistics.

The results of testing dependence by NP, KS and test RDI are presented in table 2.

Table 2. Extreme-value independency tests

PAIRED RETURN	OPTIMAL THRESHOLD*	λ_L	$\hat{\eta}$	P-VALUE	
				NP TEST	KS TEST
APPLE VER MICROSOFT	-0,0505	0,2658	1,06	0,265	0,213
VW VER PORSCHE	-0,0497	0,3026	0,95	0,027	0,034
EUR VER JPY	-0,0574	0,3729	0,55	0,179	0,071
WIG20 VER RTS	-0,0447	0,4917	1,03	0,417	0,082

* the non-parametric way of choosing the optimal threshold level estimate the Generalized Pareto Distribution parameters corresponding to various threshold levels, representing respectively 1%, 2%, 3%,.....till 12% of the extreme observations

According to tail dependence coefficients there is lower tail dependence beetween each pair of returns, whereas lower tail independence can not be rejected only for pair: VW and Porsche due to all tests.
This again shows that testing for tail independence is important since otherwise, all estimators can be biased.

There is strong difference beetwen NP, KS and test based on the residual dependence index. The last one permits also to detect the periods where tail dependence exists (on the basis on figures, look at Figure 1). It could be interesting in the future to carry out an extensive simulation study to compare the results of these tests for selected period.

7. Conclusions

Testing tail independence is simple and transparent enough to be implemented and easily monitored. Omitting the test for tail independence would introduce a large bias in the estimation and make it difficult to decide whether there is just correlation or in fact tail dependence. One important feature of this paper is the implementation of the tests for tail independence, which is recognized to be indispensable but rarely utilized in a financial context.

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