Generalized Brownian - Laplace processes and financial modeling

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Abstract

In this paper we review various stochastic processes including autoregressive processes developed recently for modeling data from financial contexts. In particular, we consider different Laplacian models and their generalizations. A general framework for Gaussian and non-Gaussian autoregressive models and their extensions is also developed and studied in detail with respect to Brownian-Laplace processes. Fractional extensions are also considered. An illustration is made with respect to a real data on exchange rates of Indian rupee and U.S. dollar.

Key words: Autoregressive processes, Brownian-Laplace processes, Financial Modeling, Fractional processes, Option pricing.

1. Introduction

In modelling financial data, a number of distributions such as the gamma, inverse Gaussian, Laplace or variance gamma, Meixner and generalized hyperbolic distributions have been used recently. Popular examples of Lévy processes are generalized hyperbolic processes and their subclasses, variance gamma processes and CGMY processes. A Lévy process based on the generalized Laplace (variance-gamma) distribution alone has no Brownian component, only linear deterministic and pure jump components. Reed (2007) and Küchler and Tappe (2008a,b) have introduced some generalizations of Laplace distribution and used them in the context of modelling option prices and stock prices data. Reed (2007) introduced the generalized normal-Laplace (GNL)(Brownian-Laplace) distribution and discussed many properties. The Black-Scholes theory of option pricing was originally based on the assumption that asset prices follow geometric Brownian motion. For such a process, the logarithmic returns ($\log(P_{t+1}/P_t)$) on the price P_t are independent and identically distributed random variables. The generalized Brownian Laplace distribution adds a Brownian component to the Lévy process. Fractional Laplace motion is used recently to model hydraulic conductivity fields in geophysics, and it may also prove useful in modeling financial time series. Hürlimann (2012) introduced an AR(1) process for inflation modeling with Brownian Laplace innovations and apply the process for modeling consumer price indices of Switzerland. They show that AR(1) model with Brownian Laplace noise has the best goodness of fit.

2. Brownian - Laplace distribution

A random variable X is said to follow the Brownian-Laplace (BL) distribution if its characteristic function (cf), is given by

$$\psi_X(s) = \left[\exp(i\nu s - \frac{\tau^2}{2}s^2)\right]\left(\frac{\alpha\beta}{(\alpha - is)(\beta + is)}\right).$$

When $\tau \to 0$, the distribution tends to an asymmetric Laplace distribution. When $\alpha \to \infty, \beta \to \infty$, it tends to a Normal distribution. When $\beta \to \infty$, the distribution exhibits fatter tail than the Normal only in the upper tail. When $\alpha \to \infty$, the distribution exhibits extra-normal variation only in the lower tail. Brownian-Laplace distribution is a mixture of convolutions of Gaussian and non-Gaussian random variables. The Brownian-Laplace distribution belongs to class $\mathcal{L}$. The Normal-Laplace is infinitely divisible and not g.i.d.. Similar to lognormal distribution we can consider log Brownian-Laplace distribution, which is also known as double Pareto-lognormal distribution. The mean and variance of the distribution are given by

$$E(X) = \nu + \frac{1}{\alpha} - \frac{1}{\beta} \text{ and } Var(X) = \tau^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2}.$$

3. Generalized Brownian-Laplace Distribution

Reed (2005) introduced the generalized Brownian-Laplace (GBL) distribution with cf

$$\varphi_X(s) = \left[\left(\exp(i\nu s - \tau^2 s^2/2)\right) \frac{\alpha\beta}{(\alpha - is)(\beta + is)} \right]^\delta.$$

This distribution arises as the convolution of independent normal and generalized Laplace distribution (Kotz et al., 2001). It is a generalization of the normal-Laplace distribution introduced by Reed and Jorgensen (2004). Hence X can be represented as

$$X \stackrel{d}{=} Z + G_1 - G_2,$$

where Z, G_1 and G_2 are independent with $Z \sim N(\delta\nu, \delta\tau^2)$ and G_1 and G_2 are independent gamma random variables with scale parameters α and β respectively and having common shape parameter δ. When $\delta = 1$, G_1 and G_2 are exponentially distributed and hence the model reduces to the normal-Laplace distribution. Lishamol and Jose (2009), Jose et al. (2008) consider the case when $\delta = 1$. When $\alpha = \beta$, $G_1 - G_2$ corresponds to the generalized Laplacian distribution. When $\alpha = \beta$, it corresponds to the generalized asymmetric Laplace distribution, also known as variance-gamma distribution (See Madan et al. (1998) and Madan and Seneta (1990)). Another generalization can be obtained by replacing δ by δ_1 and δ_2 respectively in the last two factors, in which case G_1 and G_2 are distributed as gamma but not identical. Then we get the bilateral gamma density of Küchler and Tappe (2008a).

$$E(X) = \delta \left(\nu + \frac{1}{\alpha} - \frac{1}{\beta} \right) \text{ and } Var(X) = \delta \left(\tau^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right).$$

Higher order cumulants are,

$$k_r = (r-1)!\ \delta(\alpha^{-r} + (-\beta)^{-r}), \text{ for integers } r > 2.$$

GBL distribution is infinitely divisible. GBL distribution is self-decomposable.

4. Brownian - Laplace Motion

Definition 1 *Consider a Lévy process $\{X_t\}_{t \geq 0}$ say for which the increments $X_{t+\tau} - X_\tau$ have cf $(\psi(s))^t$ where ψ is the cf of the GBL$(\nu, \tau^2, \alpha, \beta, \delta)$ distribution. Laplace motion has an infinite number of jumps in any finite time interval (a pure jump process). The extension considered here adds a continuous Brownian component to Laplace motion leading to the name Brownian-Laplace motion.*

The increments $X_{t+\tau} - X_\tau$ of this process will follow a GBL$(\nu, \tau^2, \alpha, \beta, \delta_t)$ distribution and will have fatter tails than the normal. Brownian-Laplace Motion seems to provide a good model for the movement of logarithmic prices. Option pricing for assets with logarithmic prices following Brownian-Laplace motion is discussed in Reed (2007). We consider an asset whose price S_t is given by

$$S_t = S_0 \exp(X_t)$$

where X_t is a Brownian-Laplace motion with $X_0 = 0$ and parameters $\nu, \tau^2, \alpha, \beta, \delta$.

4.1. Fractional Brownian Motion

Fractional Laplace motion is obtained by subordinating fractional Brownian motion to a gamma process. Fractional Brownian motion with parameter $H \in (0,1)$ is a centered Gaussian process $\{B_H(t), t \geq 0\}$ with $B_H(0) = 0$ and the covariance function

$$E[B_H(t)B_H(s)] = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), t, s \geq 0 \tag{1}$$

where $\sigma^2 = \text{Var}B_H(1)$. Fractional Brownian motion exhibits self-similarity with parameter H, i.e. for each $c > 0$ we have

$$\{B_H(ct), t \geq 0\} \stackrel{d}{=} \{c^H B_H(t), t \geq 0\}$$

in the sense that all finite dimensional distributions of the two processes are the same. Moreover, the stationary increment process $\{Z_k = B_H(k) - B_H(k-1)\}$, called fractional Gaussian noise, exhibits long-range dependence when $H \in (1/2, 1)$, i.e. its covariance function $\gamma(k) = E(Z_i Z_{i+k})$ tends to zero so slowly that the series $\sum_{k=1}^{\infty} \gamma(k)$ diverges, for details see Kozubowski et al. (2006).

5. Geometric Brownian-Laplace Motion

The distribution named as the double Pareto-lognormal (DPLN) distribution arises as that of the state of a geometric Brownian motion with lognormally distributed initial state, after an exponentially distributed length of time. DPLN distribution has applications in modeling earnings and income distributions, human settlement sizes, stock price returns etc. Jose and Manu (2011) developed a product autoregressive model with DPLN and log-Laplace distributions.

6. Brownian - Laplace processes

$$X_n = aX_{n-1} + \varepsilon_n;\ a \in (0,1) \text{ and } \forall\ n > 0$$

$$\psi_{X_n}(s) = \psi_{X_{n-1}}(as)\ \psi_{\varepsilon_n}(s)$$

$$\begin{aligned}\psi_\varepsilon(s) &= \frac{\psi_X(s)}{\psi_X(as)} \\ &= \left[\exp\left(i\nu(1-a)s - \frac{\tau^2(1-a^2)}{2}s^2\right)\right]\left(\frac{(\alpha - ias)}{(\alpha - is)}\frac{(\beta + ias)}{(\beta + is)}\right)\end{aligned}$$

$$\varepsilon \stackrel{d}{=} Z + E_1^* - E_2^*$$

$$Z \sim N(\nu(1-a), \tau^2(1-a^2)),\quad E_1^* \sim ET(a,\alpha) \text{ and } E_2^* \sim ET(a,\beta)$$

Brownian-Laplace model is free from zero deficiency. If $X_0 \stackrel{d}{=} BL(\nu, \tau^2, \alpha, \beta)$, then the process is strictly stationary. If X_0 is distributed arbitrarily, then also the process is asymptotically Markovian with Normal-Laplace distribution.

Theorem 1 *The AR(1) process $X_n = aX_{n-1} + \varepsilon_n$, $a \in (0,1)$ is strictly stationary Markovian with Normal-Laplace marginal distribution if and only if $\{\varepsilon_n\}$ are independently and identically distributed as $\varepsilon \stackrel{d}{=} Z + E_1^* - E_2^*$ provided $X_0 \sim BL(\nu, \tau^2, \alpha, \beta)$ and is independent of ε_1.*

7. Estimation of parameters

$$E(\varepsilon_n) = (1-a)(\nu + \frac{1}{\alpha} - \frac{1}{\beta}) \text{ and } Var(\varepsilon_n) = (1-a^2)(\tau^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2}).$$

Higher order cumulants are,

$$k_r = (r-1)!(1-a^r)(\alpha^{-r} + (-\beta)^{-r}), \text{ for integers } r > 2.$$

From the cumulants the third, fourth and fifth moments can be obtained directly since $k_3 = \mu_3$, $k_4 = \mu_4 - 3\mu_2^2$ and $k_5 = \mu_5 - 10\mu_2\mu_3$.

8. First order Generalized Brownian-Laplace Autoregressive processes

$$X_n = aX_{n-1} + \varepsilon_n; \ a \in (0,1) \text{ and } \forall\ n > 0$$

$$\begin{aligned}\varphi_\varepsilon(s) &= \left[\exp\left(i\nu\delta(1-a)s - \frac{\tau^2\delta(1-a^2)}{2}s^2\right)\right] \\ &\quad \times \left[\frac{(\alpha - ias)}{(\alpha - is)}\right]^\delta \left[\frac{(\beta + ias)}{(\beta + is)}\right]^\delta .\end{aligned}$$

Hence the distribution of ε is the convolution of Gaussian and non-Gaussian (generalized exponential tailed) random variables.

Generation of the process

This can be generated as $\epsilon = Z + G_1^* - G_2^*$, where Z follows normal and $G_i^*, i = 1,2$ are independently distributed as δ-fold convolution of exponential tailed random variables (see Jose and Manu (2011)). Expanding the cgf and solving, the moments of the sequence of innovations $\{\varepsilon_n\}$ are,

$$E(\varepsilon_n) = (1-a)\ \delta\ (\nu + \frac{1}{\alpha} - \frac{1}{\beta}) \text{ and } Var(\varepsilon_n) = (1-a^2)\ \delta\ (\tau^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2})$$

$$k_r = (r-1)!\ \delta(1-a^r)(\alpha^{-r} + (-\beta)^{-r}), \text{ for integers } r > 2.$$

Generalized Brownian-Linnik (α-Laplace) Distribution

The characteristic function has the form,

$$\phi_1(s) = \left(\exp(i\nu s - \tau^2 s^2/2)\right)^{\delta_1} \left(\frac{\lambda^\alpha}{\lambda^\alpha + |s|^\alpha}\right)^{\delta_2}.$$

We have the following special cases for different parameter values.

α	δ_1	δ_2	Distribution
2	1	1	Brownian-Laplace
2	δ	δ	Generalized Brownian-Laplace
α	1	1	Brownian α-Laplace
2	0	δ	Generalized Laplace
α	0	1	Linnik (α-Laplace)
α	0	δ	Pakes Generalized Linnik

Autoregressive models can be developed in all these cases. This establishes the wide applications of the Generalized Brownian α-Laplace distribution

9. Applications

It is the first attempt to combine Gaussian and non-Gaussian marginals to model time series data. Brownian-Laplace distribution has applications in financial modeling, Levy process, Brownian motion etc. In financial modeling, it is is a more realistic alternative for Gaussian models as logarithmic price returns do not follow exactly a Normal distribution. Exponentiated Brownian-Laplace distribution (Double Pareto-Lognormal distribution) provides a useful parametric form for modeling size distributions. Since generalized Brownian Laplace distributions possess the additive property, one can construct a Levy motion for which the increments follow the same distribution. The results are further extended to the generalized Brownian-α-Laplace distribution with a variety of applications in many areas. Thus by selecting suitable convolutions of Gaussian and non-Gaussian distributions as stationary marginal distributions of autoregressive processes we can bring Gaussian and non-Gaussian time series models to a unified framework.

10. A case study- Financial modeling

Here we consider another variant of generalized asymmetric Laplace distribution, which is a special case of generalized Brownian-Laplace distribution. In particular we consider the form considered by Küchler and Tappe (2008a,b). They refer to this as the Bilateral gamma distribution, which corresponds to the difference of two independent gamma variables so that the cf is given by

$$\psi(s) = \left[\frac{\alpha}{\alpha - is}\right]^{\delta_1} \left[\frac{\beta}{\beta + is}\right]^{\delta_2}.$$

We can consider a special case of this bilateral gamma distribution($\delta_1 = \delta_2 = \delta$) namely generalized Laplacian (GL) distribution. The cf of generalized Laplacian distribution is given by

$$\phi(t) = \frac{e^{i\theta t}}{(1 + \frac{1}{2}\sigma^2 t^2 - i\mu t)^{\delta}} \quad , \qquad -\infty < t < \infty, \quad \sigma > 0,$$

$-\infty < \mu < \infty$, where $\alpha = \frac{\sqrt{\kappa}}{\sigma}$, $\beta = \frac{\sqrt{2}}{\sigma\kappa}$. This cf can be factored as

$$\phi(t) = e^{i\theta t} \left(\frac{1}{1 + i\frac{\sigma}{\sqrt{2}}\kappa t}\right)^{\delta} \left(\frac{1}{1 - i\frac{\sigma}{\sqrt{2}\kappa} t}\right)^{\delta}, \tag{2}$$

where $\kappa > 0$, $\mu = \frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa} - \kappa\right)$. We shall use the method of moments, generalized method of moments due to Tjetjep and Seneta (2006) and method of conditional least squares due to Klimko and Nelson (1978).

Under the method of moments, the estimates can be obtained by equating the population moments and the sample moments. The resulting equations are not easily tractable. Hence we use the generalized method of moments (GMOM) due to Tjetjep and Seneta (2006).

In Conditional Least Square (CLS) method of estimation, we try to minimize the sum of squared deviations about conditional expectations. Then the conditional estimators of the parameters can be obtained by minimizing

$$E = \sum_{n=1}^{k} [x_n - E(X_n \mid X_{n-1} = x_{n-1})]^2 = \sum_{n=1}^{k} [x_n - (a x_{n-1} + (1-a)[\theta + \tau\mu])]^2.$$

Solving the normal equations, we get

$$\hat{a} = \frac{\sum_{n=1}^{k} [x_n - (\theta + \tau\mu)]\,[x_{n-1} - (\theta + \tau\mu)]}{\sum_{n=1}^{k} [x_{n-1} - (\theta + \tau\mu)]^2}.$$

$$\hat{\theta} = \frac{1}{(1-a)}\frac{1}{k}\left[\sum_{n=1}^{k} x_n - \frac{a}{\mu}\sum_{n=1}^{k} x_{n-1}\right] - \tau\mu.$$

$$\hat{\delta} = \frac{1}{(1-a)\mu}\frac{1}{k}\left[\sum_{n=1}^{k} x_n - a\sum_{n=1}^{k} x_{n-1}\right] - \frac{\theta}{\mu}.$$

$$\hat{\mu} = \frac{1}{(1-a)\tau}\frac{1}{k}\left[\sum_{n=1}^{k} x_n - a\sum_{n=1}^{k} x_{n-1}\right] - \frac{\theta}{\tau}.$$

From the data, the summary statistics obtained are as follows.

$$\begin{aligned} \text{Mean value} &= -2.6423 \times 10^{-5}, \\ \text{Variance} &= 1.2632 \times 10^{-6}, \\ \text{Coefficient of skewness} &= -0.6657. \end{aligned}$$

Now pre-fixing $\theta = 0$, the estimates of the parameters obtained by the GMOM method are,

$$\begin{aligned} \tau &= 0.10576, \\ \sigma &= 0.003447, \\ \kappa &= 1.05256. \end{aligned}$$

It is verified that these estimated values make M equal to zero, showing that the model is consistent with the data.

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