Strong laws of large numbers for capacities *

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Abstract

In this paper, with the notion of independent identically distributed random variables under sub-linear expectations initiated by Peng, we derive three kinds of strong laws of large numbers for capacities. Moreover, these theorems are natural and fairly neat extensions of the classical Kolmogorov's strong law of large numbers to the case where probability measures are no longer additive. Finally, an important feature of these strong laws of large numbers is to provide a frequentist perspective on capacities.

Keywords: capacity, strong law of large number, independently and identically distributed(IID), sub-linear expectation.

AMS 1991 subject classifications. 60H10, 60G48.

1 Introduction

The classical strong laws of large numbers (strong LLN) as fundamental limit theorems in probability theory play an important role in the development of probability theory and its applications. The key in the proofs of these limit theorems is the additivity of the probabilities and the expectations. However, such additivity assumption is not reasonable in many areas of applications because many uncertain phenomena can not be well modelled using additive probabilities or additive expectations. More specifically, motivated by some problems in mathematical economics, statistics, quantum mechanics and finance, a number of papers have used non-additive probabilities (called capacities) and nonlinear expectations (for example Choquet integral/expectation, g-expectation) to describe and interpret the phenomena (see for example, Chen and Epstein [1], Feynman [4], Gilboa [5], Huber [16], Peng [8,9], Schmeidler [17], Wakker [18], Walley and Fine [19], Wasserman and Kadane [20]). Recently, motivated by the risk measures, super-hedge pricing and modelling uncertainty in finance, Peng [11,12,13,14,15] initiated the notion of independent and identically distributed (IID) random variables under sub-linear expectations. Under this framework, he proved the weak law of large numbers and the central

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limit theorems (CLT). However, Peng's techniques can not be extended to prove the strong laws of large numbers. In this paper, we develop new approaches to solving this problem. We obtain three strong laws of large numbers for capacities in this framework. All of them are natural and fairly neat extensions of the classical Kolmogorov's strong law of large numbers, but the proofs here are different from the original proofs of the classical strong law of large numbers.

2 Results

Now we describe the problem in more detail. For a given set $\mathcal{P}$ of multiple prior probability measures on $(\Omega, \mathcal{F})$, we define a pair $(\mathbb{V}, v)$ of capacities by

$$\mathbb{V}(A) := \sup_{P\in\mathcal{P}} P(A), \quad v(A) := \inf_{P\in\mathcal{P}} P(A), \quad \forall A \in \mathcal{F}.$$

The corresponding Choquet integrals/expecations $(C_{\mathbb{V}}, C_v)$ are defined by

$$C_V[X] := \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt$$

where V is replaced by v and $\mathbb{V}$ respectively.

The pair of so-called maximum-minimum expectations $(\mathbb{E}, \mathcal{E})$ are defined by

$$\mathbb{E}[\xi] := \sup_{P\in\mathcal{P}} E_P[\xi], \quad \mathcal{E}[\xi] := \inf_{P\in\mathcal{P}} E_P[\xi].$$

Here and in the sequel, E_P denotes the classical expectation under probability P.

In general, the relation between Choquet integral and maximum -minimum expectations is as follows: For any random variable X,

$$\mathbb{E}[X] \leq C_{\mathbb{V}}[X], \qquad C_v[X] \leq \mathcal{E}[X].$$

Note that under some very special assumptions on $\mathcal{P}$ and $\mathbb{V}$, both inequalities could become equalities (see for example Gilboa [5], Huber [16], Schmeidler [17]).

Given a sequence $\{X_i\}_{i=1}^\infty$ of IID random variables for capacities, the earlier papers related to strong laws of large numbers are Dow and Werlang [2] and Walley and Fine [19]. However, the more general results for strong laws of large numbers for capacities were given by Marinacci [6,7] and Epstein and Schneider [3]. They show that, on full set, any cluster point of empirical averages lies between the lower Choquet integral $C_v[X_1]$ and the upper Choquet integral $C_{\mathbb{V}}[X_1]$ with probability one under capacity v. That is

$$v\left(\omega \in \Omega : C_v[X_1] \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i(\omega) \leq C_{\mathbb{V}}[X_1]\right) = 1.$$

Marinacci [6,7] obtains his result under the assumptions that $\mathbb{V}$ is a totally monotone capacity on a Polish space Ω, random variables $\{X_n\}_{n\geq1}$ are bounded or continuous. Epstein and Schneider [3] also show the same result under the assumptions that $\mathbb{V}$ is rectangular and the set $\mathcal{P}$ is finite.

Since the gap between the Choquet integrals $C_v[X]$ and $C_V[X]$ is bigger than that of the maximum-minimum expectations $\mathcal{E}[X]$ and $\mathbb{E}[X]$ for all X, it is of interest to ask whether we can obtain a more precise result if the Choquet integrals/expctations in the above equality are replaced by maximum-minimum expectations. That is

$$v\left(\omega\in\Omega:\mathcal{E}[X_1]\le\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i(\omega)\le\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i(\omega)\le\mathbb{E}[X_1]\right)=1.$$

The first result in this paper is to show that the above equality is still true in Peng's framework even under some weaker assumptions. Furthermore, motivated by this result, we establish two new laws of large numbers. The first is to show that there exist two cluster points of empirical averages which reach the minimum expectation $\mathcal{E}[X_1]$ and the maximum expectation $\mathbb{E}[X_1]$ respectively under capacity $\mathbb{V}$. That is

$$\mathbb{V}\left(\omega\in\Omega:\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i(\omega)=\mathbb{E}[X_1]\right)=1;$$

$$\mathbb{V}\left(\omega\in\Omega:\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i(\omega)=\mathcal{E}[X_1]\right)=1.$$

The second is to prove that the cluster set of empirical averages coincides with the interval between minimum expectation $\mathcal{E}[X_1]$ and maximum expectation $\mathbb{E}[X_1]$. That is, if $C(\{x_n\})$ is the cluster set of $\{x_n\}$, then

$$\mathbb{V}\left(\omega\in\Omega:C\left(\left\{\frac{1}{n}\sum_{i=1}^{n}X_i(\omega)\right\}\right)\supseteq[\mathcal{E}[X_1],\ \ \mathbb{E}[X_1]]\right)=1.$$

Obviously, if capacity $\mathbb{V}$ or v in the above results is a probability measure, all of our main results are natural and fairly neat extension of the classical Kolmogorov's strong law of large numbers. Moreover, an important feature of our strong laws of large numbers is to provide a frequentist perspective on capacities.

Finally, our results also imply that $[\mathcal{E}[X_1],\mathbb{E}[X_1]]$ is the smallest interval of all intervals in which the limit point of empirical averages lies with probability (capacity v) one.

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