Critical two-point functions for long-range self-avoiding walk in high dimensions

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March 14, 2013

Abstract

We consider long-range self-avoiding walk, percolation and the Ising model on $\mathbb{Z}^d$ that are defined by power-law decaying pair potentials of the form $D(x) \asymp |x|^{-d-\alpha}$ with $\alpha > 0$. The upper-critical dimension $d_{\rm c}$ is $2(\alpha\wedge 2)$ for self-avoiding walk and the Ising model, and $3(\alpha\wedge 2)$ for percolation. Let $\alpha \neq 2$ and assume certain heat-kernel bounds on the n-step distribution of the underlying random walk. We prove that, for $d > d_{\rm c}$ (and the spread-out parameter sufficiently large), the critical two-point function $G_{p_{\rm c}}(x)$ for each model is asymptotically $C|x|^{\alpha\wedge 2-d}$, where the constant $C \in (0,\infty)$ is expressed in terms of the model-dependent lace-expansion coefficients and exhibits crossover between $\alpha < 2$ and $\alpha > 2$. We also provide a class of random walks that satisfy those heat-kernel bounds.

1 Introduction

Self-avoiding walk is a model for linear polymers. We define the two-point function for SAW on $\mathbb{Z}^d$ as

$$G_p^{\rm SAW}(x) = \sum_{\omega:o\to x} p^{|\omega|} \prod_{j=1}^{|\omega|} D(\omega_j - \omega_{j-1}) \prod_{s<t}(1-\delta_{\omega_s,\omega_t}), \tag{1.1}$$

where $p \geq 0$ is the fugacity, $|\omega|$ is the length of a path $\omega = (\omega_0, \omega_1, \dots, \omega_{|\omega|})$ and $D : \mathbb{Z}^d \to [0,1]$ is the $\mathbb{Z}^d$-symmetric non-degenerate (i.e., $D(o) \neq 1$) 1-step distribution for the underlying random walk; the contribution from the 0-step walk is considered to be $\delta_{o,x}$ by convention. If the indicator function $\prod_{s<t}(1-\delta_{\omega_s,\omega_t})$ is replaced by 1, then $G_p^{\rm SAW}(x)$ turns into the RW Green's function $G_p^{\rm RW}(x)$, whose radius of convergence $p_{\rm c}^{\rm RW}$ is 1, as $\chi_p^{\rm RW} \equiv \sum_{x\in\mathbb{Z}^d} G_p^{\rm RW}(x) = (1-p)^{-1}$ for $p < 1$ and $\chi_p^{\rm RW} = \infty$ for $p \geq 1$. Therefore, the radius of convergence $p_{\rm c}^{\rm SAW}$ for $G_p^{\rm SAW}(x)$ is not less than 1. It is known that $\chi_p^{\rm SAW} \equiv \sum_{x\in\mathbb{Z}^d} G_p^{\rm SAW}(x) < \infty$ if and only if $p < p_{\rm c}^{\rm SAW}$ and diverges as $p \uparrow p_{\rm c}^{\rm SAW}$.

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We are interested in asymptotic behavior of $G_{p_c}(x)$ as $|x| \to \infty$. For the "uniformly spread-out" finite-range models, e.g., $D(x) = \mathbb{1}_{\{|x|=1\}}/(2d)$ or $D(x) = \mathbb{1}_{\{\|x\|_\infty \le L\}}/(2L+1)^d$ for some $L \in [1, \infty)$, it has been proved [3, 4, 6] that, if $d > 4$ and L is sufficiently large, then there is a model-dependent constant A ($= 1$ for random walk) such that

$$G_{p_c}(x) \underset{|x|\to\infty}{\sim} \frac{a_d/\sigma^2}{A|x|^{d-2}}, \tag{1.2}$$

where "$\sim$" means that the asymptotic ratio of the left-hand side to the right-hand side is 1, and

$$a_d = \frac{d\Gamma(\frac{d-2}{2})}{2\pi^{d/2}}, \qquad \sigma^2 \equiv \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) = O(L^2). \tag{1.3}$$

In this paper, we investigate long-range self-avoiding walk on $\mathbb{Z}^d$ defined by power-law decaying pair potentials of the form $D(x) \asymp |x|^{-d-\alpha}$ with $\alpha > 0$. For example, we can consider the following uniformly spread-out long-range D with parameter $L \in [1, \infty)$:

$$D(x) = \mathcal{N}_L \|\tfrac{x}{L}\|_1^{-d-\alpha}\big(1 + O(\|\tfrac{x}{L}\|_1^{-\varepsilon})\big), \tag{1.4}$$

for some $\varepsilon > 0$, where $\mathcal{N}_L = O(L^{-d})$ is the normalization constant and $\|x\|_\ell = |x| \vee \ell$. As a result,

$$D(x) = O(L^\alpha)\|x\|_L^{-d-\alpha}, \tag{1.5}$$

which we require throughout the paper (cf., Assumption 1.1 below). The goal is to see how the asymptotic expression (1.2) of $G_{p_c}(x)$ changes depending on the value of α.

It has been proved [5] that, for $d > d_c := 2(\alpha \wedge 2)$ and $L \gg 1$, the Fourier transform $\hat{G}_p(k) \equiv \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} G_p(x)$ for the long-range models is bounded above and below by a multiple of $\hat{G}^{\rm RW}_{\hat{p}}(k) \equiv (1 - \hat{p}\hat{D}(k))^{-1}$ with $\hat{p} = p/p_c$, uniformly in $p < p_c$. Although this gives an impression of the similarity between $G_{p_c}(x)$ and $G^{\rm RW}_1(x)$, it is still too weak to identify the asymptotic expression of $G_{p_c}(x)$. The proof of the above Fourier-space result makes use of the following properties of D that we make use of here as well: there are $v_\alpha = O(L^{\alpha\wedge 2})$ and $\epsilon > 0$ such that

$$\hat{D}(k) \equiv \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} D(x) = 1 - v_\alpha |k|^{\alpha\wedge 2} \times \begin{cases} 1 + O((L|k|)^\epsilon) & [\alpha \ne 2], \\ \log \frac{1}{L|k|} + O(1) & [\alpha = 2]. \end{cases} \tag{1.6}$$

If $\alpha > 2$, then $v_\alpha = \sigma^2/(2d)$. Moreover, if $L \gg 1$, there is a constant $\Delta \in (0, 1)$ such that

$$\|D^{*n}\|_\infty \le O(L^{-d})\, n^{-\frac{d}{\alpha\wedge 2}} \quad [n \ge 1], \qquad 1 - \hat{D}(k) \begin{cases} < 2 - \Delta & [k \in [-\pi, \pi]^d], \\ > \Delta & [\|k\|_\infty \ge L^{-1}]. \end{cases} \tag{1.7}$$

1.1 Main result

In addition to the above properties, the n-step transition probability obeys the following bound:

$$D^{*n}(x) \le \frac{O(L^{\alpha\wedge 2})}{\|x\|_L^{d+\alpha\wedge 2}}\, n \times \begin{cases} 1 & [\alpha \ne 2], \\ \log \|x\|_L & [\alpha = 2]. \end{cases} \tag{1.8}$$

To overcome this difficulty, we assume the following bound on the discrete derivative of the n-step transition probability:

$$\left|D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2}\right| \le \frac{O(L^{\alpha\wedge 2})\|\!|y|\!\|_L^2}{\|\!|x|\!\|_L^{d+\alpha\wedge 2+2}}\, n \qquad [|y| \le \tfrac{1}{3}|x|]. \tag{1.9}$$

Here is the summary of the required properties of D.

Assumption 1.1. *The $\mathbb{Z}^d$-symmetric 1-step distribution D satisfies the properties (1.5), (1.6), (1.7), (1.8) and (1.9).*

Under the above assumption on D, we can prove the following theorem:

Theorem 1.2. *Let $\alpha > 0$, $\alpha \neq 2$ and*

$$\gamma_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}, \tag{1.10}$$

and assume all properties of D in Assumption 1.1. Then, for random walk with $d > \alpha\wedge 2$ and any $L \ge 1$, and for self-aboiding walk with $d > d_c$ and $L \gg 1$, there are $\mu \in (0, \alpha\wedge 2)$ and $A = A(\alpha, d, L) \in (0, \infty)$ ($A \equiv 1$ for random walk) such that, as $|x| \to \infty$,

$$G_{p_c}(x) = \frac{\gamma_\alpha/v_\alpha}{A|x|^{d-\alpha\wedge 2}} + \frac{O(L^{-\alpha\wedge 2+\mu})}{|x|^{d-\alpha\wedge 2+\mu}}. \tag{1.11}$$

Moreover, p_c and A can be expressed in term of Π_p as

$$p_c = \hat{\Pi}_{p_c}(0)^{-1}, \qquad A = p_c + \begin{cases} 0 & [\alpha < 2], \\ \dfrac{p_c^2}{\sigma^2}\displaystyle\sum_x |x|^2 \Pi_{p_c}(x) & [\alpha > 2]. \end{cases} \tag{1.12}$$

1.2 Conclusion

(a) The goal of this paper is to overcome those difficulties and derive an asymptotic expression of the critical two-point function for the power-law decaying long-range models above the critical dimension, using the lace expansion.

(b) The finite-range models are formally considered as the $\alpha = \infty$ model. Indeed, the leading term in (1.11) for $\alpha > 2$ is identical to (1.2).

(c) As described in (1.12), the constant A exhibits crossover between $\alpha < 2$ and $\alpha > 2$; in particular, $A = p_c$ for $\alpha < 2$. According to some rough computation, it seems that the asymptotic expression of $G_{p_c}(x)$ for $\alpha = 2$ is a mixture of those for $\alpha < 2$ and $\alpha > 2$, with a logarithmic correction:

$$G_{p_c}(x) \underset{|x|\to\infty}{\sim} \frac{\gamma_2/v_2}{p_c|x|^{d-2}\log|x|}. \tag{1.13}$$

One of the obstacles to prove this conjecture is a lack of good control on convolutions of the random walk Green's function and the lace-expansion coefficients for $\alpha = 2$. As hinted in the above expression, we may have to deal with logarithmic factors more actively than ever.

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