Branching Random Walks and Their Applications to Population Studies

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Abstract

Recent investigations have demonstrated that continuous-time branching random walks on multidimensional lattices give an important example of stochastic models in which the evolutionary processes depend on the structure of a medium and the spatial dynamics. It is convenient to describe such processes in terms of birth, death, and walks of particles on the lattice. The structure of a medium is defined by the offspring reproduction law at a finite number of generation centers situated on the lattice points. We consider models of branching random walks under different assumptions about underlying random walks: symmetric or non-symmetric, with or without the finite variance of jumps. The goal of the study is to analyze phase transitions for various models of branching random walks. We start by the classification of branching random walks depending on properties of the underlying random walks and the lattice dimension. Limit theorems for the numbers of particles at an arbitrary point of the lattice and for the particle population size are obtained. For investigation of the population front of particles the large deviation for branching random walks are studied.

Keywords: non-homogeneous environments, limit theorems, large deviations.

1 Introduction

Consider particles living independently of each other and of their history. Each particle walks on the lattice $\mathbf{Z}^d$ until it reaches a source where its behavior changed. Branching sources are of three types, depending on whether branching or violation of symmetry of the walk takes place or not. In sources of the first type, particles die or are born with keeping the random walk symmetry (Albeverio et al., 1998; Bogachev and Yarovaya, 1998a,b; Yarovaya, 2007). In sources of the second type, walk symmetry is violated by increasing the degree of dominance of branching or walk Vatutin et al. (2003); Vatutin and Topchii (2004); Yarovaya (2010). Sources of the third type should be called "pseudo-sources," because in them only the walk symmetry, without birth or death of particles, is violated. BRWs with r sources of the first type, k of the second type, and m of the third type are denoted BRW$/r/k/m$ Yarovaya (2013).

In Sec. 2, the asymptotic results depending on the lattice dimension for BRW$/1/0/0$ with finite variance of jumps are presented. In particular asymptotic behavior of survival probabilities and limit theorems for the numbers of particles, both at an arbitrary point of the lattice and on the entire lattice, are obtained. After that in Sec. 2 we consider the effects for BRW$/1/0/0$ in another case, when corresponding transition rates of the random walk have heavy tails. As a result, the variance of the jumps is infinite, and a random walk may be transient even on one- or two-dimensional lattices. Conditions of transience for a random walk and limit theorems for the numbers of particles are obtained.

In Sec. 3, we present the results about BRW/0/1/0 with violation of symmetry of the random walk at a source. Moreover, we give some results for the general model BRW/r/k/m of a BRW with a finitely many branching sources. In a supercritical case for such processes the phase transitions are discovered. This situation differs substantially from the BRWs with a single source.

In Sec. 4, the behavior of transition probabilities of BRW/1/0/0 with finite variance of jumps in the situation when the space and time variables grow jointly are established. One of the main results here is the limit theorem about limiting properties of the Green function for the transition probabilities. These results are important for the investigation of the large deviations for branching random walks, in particular, for studying of the particle population front.

2 Symmetric BRWs with a single source

Let us consider a BRW with a the matrix $A = \|a(x,y)\|_{x,y\in\mathbf{Z}^d}$ of the random walk transition intensities: $a(x,y) \geq 0$ for $x \neq y$, $a(x,x) < 0$, and $\sum_y a(x,y) = 0$; $a(x,y) = a(y,x)$; $a(x,y) = a(0,y-x) = a(y-x)$. Let us also suppose that the random walk is irreducible. We assume that the branching takes place only at the origin and is determined by the infinitesimal generating function $f(u) := \sum_{n=0}^{\infty} b_n u^n$, where $b_n \geq 0$ for $n \neq 1$, $b_1 < 0$ and $\sum_n b_n = 0$. We assume also that $\beta_r = f^{(r)}(1) < \infty$, $r \in \mathbf{N}$, $\beta = \beta_1 = f'(1)$, where $f^{(r)}$ stands for the rth derivative of f. Denote by $\mu_t(y)$ the random number of particles at site $y \in \mathbf{Z}^d$ and by $\mu_t := \sum_y \mu_t(y)$ the total population size at time $t \geq 0$. We shall be concerned with the statistical moments $m_n(t,x,y) := \mathsf{E}_x \mu_t^n(y)$ and $m_n(t,x) := \mathsf{E}_x \mu_t^n$ $(n \in \mathbf{N})$, where E_x stands for the expectation under the condition $\mu_0(\cdot) = \delta_x(\cdot)$ (we assume there is initially a single particle located at x). Therefore, if $\mu_t(0) > 0$ particles were at time t at the origin, then, independently of the rest of particles, each particle in the time interval $[t, t+h)$ can either jump with the probability $p(h,0,y) = a(y)h + o(h)$ to a point $y \neq 0$, or produce n particles $(n \neq 1)$ including itself, or die (the case of $n = 0$) with the probability $p_*(h,n) = b_n h + o(h)$, or survive (no changes) with the probability $1 - \sum_{y\neq 0} a(y)h - \sum_{n\neq 1} b_n h + o(h)$. By standard argumentation, the sojourn time of a particle at the origin is exponentially distributed with the parameter $-(a(0)+b_1)$. Transtion probabilities $p(t,x,y)$ satisfies the system of Kolmogorov's backward equations $\frac{\partial p}{\partial t} = Ap$, $p(0,x,y) = \delta_y(x)$, where the (linear) operator A acts with respect to variable x, that is, $Ap(t,x,y) := \sum_{x'} a(x,x')p(t,x',y)$, while y is treated as a parameter. Denote by $G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t,x,y)\,dt$ the Green function, that is the Laplace transform of the transition probability. Put $G_0 = G_0(0,0)$. A random walk will be called transient if $G_0 < \infty$ and recurrent if $G_0 = \infty$.

BRW/1/0/0 with finite variance of jumps. *If we consider $a(\cdot)$ under the condition $\sum_{z\in\mathbf{Z}^d} |z|^2 a(z) < \infty$, which implies finite variance of jumps for a random walk, then $G_0 < \infty$ for $d \geq 3$ and $G_0 = \infty$ for $d = 1, 2$.*

BRW/1/0/0 with heavy tails. *Given a continuous positive function $H : [-\pi,\pi]^d \to \mathbf{R}$ satisfying $H(x) = H(-x)$, for all $x \in [-\pi,\pi]^d$, denote by H_0 and H^0 the positive constants such that $0 < H_0 \leq H(x) \leq H^0$, for all $x \in [-\pi,\pi]^d$. Let us suppose that the function $a(\cdot)$ satisfies the condition $a(z) = \frac{H\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}$, where*

$\alpha \in (0,2)$, $z \in \mathbf{Z}^d$ and $z \neq 0$. In this case $G_0 < \infty$ for $d = 1$ and any $0 < \alpha < 1$, and for $d \geq 2$ and any $0 < \alpha < 2$.

Consider now the basic equation $\beta G_\lambda(0,0) = 1$ (with respect to λ) for both models. Note that the function $G_\lambda(0,0)$ is strictly decreasing on $(0,\infty)$ and $G_\lambda(0,0) \downarrow 0$ as $\lambda \uparrow \infty$. Therefore, the equation $\beta G_\lambda(0,0) = 1$ has a (unique) positive root $\lambda_0 = \lambda_0(\beta)$ whenever $\beta > \beta_c$, where

$$\beta_c := \begin{cases} 1/G_0(0,0) & \text{if} \quad G_0(0,0) < \infty, \\ 0 & \text{if} \quad G_0(0,0) = \infty. \end{cases}$$

As will be shown below, $\beta = \beta_c$ is a critical point for the BRW under consideration, and the case $\beta > \beta_c$ corresponds to the supercritical regime wherein the parameter $\lambda_0 > 0$ determines the rate of the process exponential growth.

The main result consists in that the following theorem obtained earlier for the BRW with finite variance of jumps may be extended to BRW with heavy tails and shows that the asymptotic behavior of BRW/1/0/0 is different according to $\beta > \beta_c$, $\beta = \beta_c$ or $\beta < \beta_c$ in both cases. If $\beta > \beta_c$ then the moments m_n, $n = 1, 2, \ldots$, as $t \to \infty$, have the asymptotics

$$m_n(t,x,y) \sim C_n(x,y)\, e^{n\lambda_0 t}, \qquad m_n(t,x) \sim C_n(x)\, e^{n\lambda_0 t},$$

where the constants $C_n(x,y)$, $C_n(x)$ are determined recursively. By this statement, the numbers of particles $\mu_t(y)$ and $\mu_t = \sum_{y\in\mathbf{Z}^d} \mu_t(y)$ have some limiting distributions when $t \to \infty$, under the normalization $e^{-\lambda_0 t}$.

Moreover, one of the principal effects of BRW on $\mathbf{Z}^d$ with heavy tails is the existence of the nontrivial critical value $\beta_c > 0$ for all dimensions d of the lattice which contrasts with the case of BRW with finite variance of jumps, where the nontrivial critical value $\beta_c > 0$ exists only for $d \geq 3$.

3 Multi-source BRWs

In BRW/0/1/0 the random walk of particles is governed by the matrix of transition intensities $\bar{A} = (\bar{a}(x,y))_{x,y\in\mathbb{Z}^d}$, where $\bar{a}(x,y) = a(x-y)$ only for $x \neq x_0$ and all y. At the source $x = x_0$, we have $\bar{a}(x_0,y) = -(1-\alpha)\frac{a(x_0-y)}{a(0)}$ with a parameter $\alpha \in (0,1)$ for any $y \in \mathbb{Z}^d$. So, in contrast to the case of BRW/1/0/0, the matrix $\bar{A}$ is nonsymmetric. Branching occurs at the source x_0 and is defined by the infinitesimal generating function $\bar{f}(u) = \sum_{n=0}^{\infty} \bar{b}_n u^n$. The sojourn time of a particle at the source is distributed exponentially with the parameter $-(\bar{b}_1 + \alpha - 1)$. If $\bar{b}_n = \alpha b_n$ for all $n \geq 0$ then $\bar{f}(u) \equiv f(u)$, where $f(u)$ is the generating function of BRW/1/0/0.

In BRW/r/k/m with a finite number of sources, more general multi-point perturbations of the generator of the symmetric random walk A are used than in BRW with a single source of branching. This follows from the statment that the mean numbers of particles $m_1(t) = m_1(t,\cdot,y)$ in BRW/r/k/m are governed by:

$$m'(t) = Y m_1(t), \quad m_1(0) = \delta_y, \tag{1}$$

where

$$Y = A + \left(\sum_{s=1}^{r} \beta_s \Delta_{z_s}\right) + \left(\sum_{i=1}^{k} \zeta_i \Delta_{x_i} A + \sum_{i=1}^{k} \eta_i \Delta_{x_i}\right) + \left(\sum_{j=1}^{m} \chi_j \Delta_{y_j} A\right). \tag{2}$$

Here, $A : l^p(\mathbb{Z}^d) \to l^p(\mathbb{Z}^d)$, $p \in [1,\infty]$, is a symmetric operator, $\Delta_x = \delta_x \delta_x^T$, and $\delta_x = \delta_x(\cdot)$ denotes a column-vector on the lattice taking the unit value at the point x and vanishing at other points. The same equation is also valid for $m_1(t) = m_1(t,\cdot)$ with the initial condition $m_1(0) = 1$ in $l^\infty(\mathbb{Z}^d)$. The operator (2) can be written as

$$Y = A + \sum_{i=1}^{k+m} \zeta_i \Delta_{u_i} A + \sum_{j=1}^{k+r} \beta_j \Delta_{v_j}.$$

In each of the sets $U = \{u_i\}_{i=1}^{k+m}$, and $V = \{v_j\}_{j=1}^{k+r}$, the points are pairwise distinct, but U and V may have a nonempty intersection. The points from $V \setminus U$ correspond to r sources of the first type; those from $U \cap V$ to k sources of the second type; and those from $U \setminus V$ to m sources of the third type.

In Yarovaya (2013), the problem of calculation of the eigenvalues of the operator Y is reduced to the problem of finding the values of λ under which the determinant of some auxiliary finite-dimensional linear system vanishes. The main problem of determining whether these eigenvalues λ of Y are positive or not is still to be resolved in a general case of multiple sources. Yarovaya (2010) provided conditions for λ to be positive for BRW with a single source. This problem is solved for some special cases: BRW2/0/0, BRW0/2/0, etc.

The condition that the highest positive eigenvalue λ_0 of the operator Y is strictly positive guarantees the exponential growth of the first moment of the total number of particles both at an arbitrary point and on the entire lattice Yarovaya (2010). As in BRW/1/0/0 and BRW/0/1/0, the same condition $\lambda_0 > 0$ implies the exponential growth of the higher-order moments. For all $n \in \mathbb{N}$ and $x, y \in \mathbb{Z}^d$, if:

$$m(n,x,y) = \lim_{t\to\infty} \frac{E_x \mu_t^n(y)}{m_1^n(t,x,y)} = \lim_{t\to\infty} \frac{m_n(t,x,y)}{m_1^n(t,x,y)},$$
$$m(n,x) = \lim_{t\to\infty} \frac{E_x \mu_t^n}{m_1^n(t,x)} = \lim_{t\to\infty} \frac{m_n(t,x)}{m_1^n(t,x)},$$

then, the limits

$$\lim_{t\to\infty} \mu_t(y)\, e^{-\lambda_0 t} = \xi \psi(y), \qquad \lim_{t\to\infty} \mu_t\, e^{-\lambda_0 t} = \xi, \tag{3}$$

where $\psi(y)$ is the eigenfunction corresponding to the eigenvalue λ_0 and ξ is a nondegenerate random variable, are valid for multiple sources in the sense of moment convergence. Eq. (3) reflects the exponential growth of the total number of particles both at an arbitrary point and on the entire lattice with the parameter λ_0. Additionally, by the Carleman criterion, if the growth rate $m(n,x)$ is limited by the condition $\sum_{n=1}^{\infty} m(n,x)^{-1/(2n)} = \infty$, then the moments define the distribution ξ uniquely. In this case, relations Eq. (3) are valid in the sense of convergence in distribution, too.

The recent investigations of cellular dynamics have demonstrated the importance of development of stochastic models in which the evolutionary processes depend on the structure of a medium. The model of the BRW/r/k/m can be used for description of the evolution of cell populations with migration of cells. One of the main applied problem arising here is an investigation of limit distributions of the cell population. Such problems may be solved in the frame of BRW/r/k/m, and the limit distributions for the numbers of cells both at an arbitrary point and on the entire lattice can be obtained.

4 Spatio-temporal stucture of BRWs

We consider a continuous-time BRW$r/0/0$ where the particles can reproduce one counterpart at a few lattice points (the sources of branching) with large deviations for a random walk with the evolutionary operator

$$Y = A + \sum_{s=1}^{r} \beta_s \Delta z_s.$$

The random walk generator is defined, as in Sec. 2, by the operator

$$(Af)(x) = \sum_{x'} a(x - x')f(x'),$$

in the space $l^p(\mathbf{Z}^d)$, $1 \le p \le \infty$, where $a(x) = a(-x)$ and $\sum_{z \in \mathbf{Z}^d} |z|^2 a(z) < \infty$, therefore $A = A^*$ in $l^2(\mathbf{Z}^d)$. We assume also that $\sum_{x \neq 0} a(x) = -a(0) = 1$ and, for $\eta \in \mathbf{R}^d$, the series $\sum_{x \neq 0} a(x)(e^{(\eta,z)} - 1)$ converges.

Molchanov and Yarovaya (2012a,b) obtained limit theorems for transition probabilities $p(t, x, y)$ as $|y - x| \to \infty$, satisfying $\partial_t p = Ap$ with the initial conditions $p(0, x, y) = \delta_y(x)$, and their Green functions $G_\lambda(x, y)$. In particular, they showed that for every fixed $\lambda > 0$ the Green function $G_\lambda(x, y)$ has the asymptotic representation

$$G_\lambda(x, y) \asymp \frac{e^{-|y-x| h_\lambda (y-x/|y-x|)}}{|y - x|^{\frac{d-1}{2}}} r_\lambda(y - x/|y - x|), \quad |y - x| \to \infty,$$

where functions h_λ and r_λ are defined by the characteristics of BRW.

The spectrum of the evolution operator Y generating the mean numbers of particles consists of two components: the pure absolutely continuous spectrum $\sigma_{ac}(Y) \subset [-2, 0]$ and the discrete spectrum $\sigma_d(Y)$, containing at most r non-negative eigenvalues. If $d = 1, 2$ (the recurrent case) then $\sigma_d(Y) \neq \varnothing$ for any $\beta > 0$. If $d \ge 3$ then one can find $\beta_c > 0$ such that $\sigma_d(Y) = \varnothing$ for $\beta < \beta_c$ and $\sigma_d(Y) \neq \varnothing$ for $\beta > \beta_c$. Special attention is paid to the case when $\operatorname{card} \sigma_d(Y) = 1$ and the relative eigenvalue $\lambda_0(\beta)$ is positive. If $\beta > \beta_c$ then for every fixed $\eta > 0$ the asymptotic behavior of the eigenfunction $\psi_0(x, \beta)$ has the representation: $\psi_0(x, \beta) \asymp G_{\lambda_0(\beta)}(0, x)$. If $\beta \to \beta_c$ for $d \ge 3$ or $\beta \to 0$ for $d = 1, 2$ then $\lambda_0(\beta) \to 0$, and the expression for the eigenfunction is obtained by Molchanov and Yarovaya (2012b) in the explicit form:

$$\psi_0(x, \beta) \asymp e^{-\sqrt{\lambda_0(\beta)}|x|(1+o(1))}.$$

Put $x = 0$ and $m_1(t, 0, y) = \mathsf{E}_0 \mu_t(y)$. Let there exist β_1 such that for $\beta_c < \beta < \beta_1$ the spectrum $\sigma_d(Y)$ has a unique eigenvalue $\lambda_0(\beta) > 0$. The moment $m_1(t, 0, y)$ satisfies Eq. (1). Then $m_1(t, 0, y) = e^{\lambda_0(\beta)t} \psi_0(y) \psi_0(0) + v(t, y)$, where $\|\psi_0\|_2 = 1$, and $v(t, \cdot)$ is the function uniformly bounded for all t in $l^2(\mathbf{Z}^d)$.

Assume, that $\mu(0, 0, y) = \delta_0(y)$ and $\beta_c < \beta < \beta_1$. The set

$$\Gamma_t = \{y : \mathsf{E}_0 \mu_t(y) = m_1(t, 0, y) \le C\}$$

is called *the front of population.*

Theorem 1 *Let us* $\ln G_\lambda(0, y) \sim -|y| r_{\lambda,\varkappa}\left(\frac{y}{|y|}\right)$, *as* $|y| \to \infty$. *Then*

$$\Gamma_t = \left\{ y : \frac{|y|}{t} r_{\lambda_0(\beta),\varkappa}\left(\frac{y}{|y|}\right) \ge \lambda_0(\beta) + o(1) \right\},$$

as $|y| \to \infty$ *and* $t \to \infty$ *such that* $|y| = O(t)$.

As follows from the asymptotic representation for $G_\lambda(0, y)$, as $\lambda \ll 1$, for $\beta \to \beta_c$ the front approximately has the spherical form: $\Gamma_t \approx \left\{ y : |y| \geq t\sqrt{\lambda_0(\beta)} \right\}$ and extends linearly with time. The topic is discussed by Molchanov and Yarovaya (2012c) with the analysis of the population for $\beta < \beta_c$ and for $\beta \to \beta_c$.

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