Recursively Generated Control Theoretic Splines for On-the-Fly Approximation

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Abstract

When data is received in small batches it is useful to be able to construct preliminary spline approximations using the data as it is received. In this paper we will present a constructive method for producing splines in a recursive manner. The method is based on using the previous spline as a target curve and modifying it based on the newly created data. This also allows the dimension of critical matrices to be kept small which increases numerical stability.

Keywords: minimum norm problem, tracking, real time

1 Introduction

Splines, both smoothing and interpolating, are ubiquitous in all problems in which it is required to construct from data a curve. However for large data sets the direct methods of construction involve solving large systems of linear equations and or inverting large matrices. In this paper we investigate the problem of treating large data sets in a recursive manner in order to keep the dimensions of problems to be solved under a fixed size.

The particular problem that motivated this line of work was boundary reconstruction when the boundary is being measured at isolated points using a remote device. It is assumed that the boundary was closed and that arbitrarily many measurements could be made. It was assumed that the device could make a series of measurements in one complete revolution and that that additional revolutions could made. Thus it was hypothesized that N measurement were made at each revolution and a smoothing spline was constructed after the first revolution was complete. After the second revolution the second set of data was to be used to modify the first smoothing spline and so forth. The idea is very similar to the problem of using new data to update an existing map. Results of this problem have been reported in [6, 7, 8]. Recently a similar problem has been considered by Han-Fu Chen, [1].

In this paper we consider the general problem and show that there are effective ways to attack the general of constructing recursive control theoretic splines. We drop the assumption of a closed curve and simply assume that we are gathering data in time from an unknown function. We assume that there is an unknown function $f(t)$ and that we are sampling it in time with noisy measurements. Our data set is of the form $\{(t_i, f(t_i) + \epsilon_i) :\ t_1 < t_2 < \cdots\}$.

2 Statement of the basic problem

Let

$$\dot{x} = Ax + bu, \quad y = cx, \quad x(0) = x_0 \tag{1}$$

be a controllable and observable linear system with $x \in R^m$ be the spline generator. We will follow the notation and definitions of [4]. We assume a sequence of data sets, D_n, of equal size

$$D_n = \{(t_i, \alpha_i) : \; i = n-k+1, \cdots, n\}. \tag{2}$$

We further assume that the data is of the form

$$\alpha_i = f(t_i) + \epsilon_i$$

where $f(t)$ is a continuous function that is at least piecewise smooth and the ϵ's are values of an iid random variable that is at least symmetrically distributed about 0. These conditions have been studied in [3] and [11]. We assume that the set

$$Time = \{t_i : \; t_1 < t_2 < \cdots\}.$$

For $n = 1, 2, \cdots$ we let the cost function be given by

$$J_n(u) = \int_0^{t_n} u(t)^2 dt + \lambda_n \int_0^{t_{n-1}} (y(t) - y_{n-1}(t))^2 dt + \sum_{i=n-k+1}^{n} (y(t_i) - \alpha_i)^2 \tag{3}$$

We let $u_n(t)$ and $y_n(t)$ be the optimal control and resulting output with respect to this cost function. Here the idea is that we are encoding the past data as the spline function $y_{n-1}(t)$. The coefficients λ_n form a sequence of numbers that approach infinity. As the λs become large either the values of the cost function will become unbounded or the sequence of optimal spline functions $\{y_n(t)\}$ will converge. One of the main goals of the paper is to show that the sequence of λs can be chosen in such a way that the sequence of smoothing splines $\{y_n(t)\}_{n=1}^{\infty}$ converges.

To solve this problem we formulate it as a minimum norm problem in Hilbert space. We follow closely the development in [12]. We let

$$H_n = \{(u, h, \alpha) : \; (u, h, \alpha) \in L_2[0, t_n] \times L_2[0, t_{n-1}] \times R^k\}$$

with norm

$$\|(u, h, \alpha)\|^2 = \int_0^{t_n} u(t)^2 dt + \lambda_n \int_0^{t_{n-1}} h(t)^2 dt + \alpha' \alpha.$$

As in [12] let

$$\ell_i(s) = \begin{cases} ce^{A(t_i - s)} b & t_i - s \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\ell(s) = \begin{cases} ce^{A(t - s)} b & t - s \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

We now define a closed linear affine variety

$$V_{x_0} = \{(u, y, z) : \; y(t) = ce^{At} x_0 + \int_0^{t_n} \ell(s) u(s) ds, \quad z_i = ce^{At_i} x_0 + \int_0^{t_n} \ell_i(s) u(s) ds\} \tag{4}$$

We define the data point in H to be the point

$$p_n = (0, y_{n-1}(t), \alpha^n) \tag{5}$$

where

$$\alpha^n = (\alpha_{n-k+1}, \cdots, \alpha_n).$$

The optimization problem is now to find the unique point in the linear variety V_{x_0} that is closest to the data point p_n. We follow the algorithm of [12]. Since the data is continuous this results in solving a system of linear integral equations. These may be solved by iteration. In the next section we will reformulate the basic problem by approximating the second integral.

In [3] we studied the following problem: Let

$$J^N(u) = \int_0^T u(t)^2 dt + \sum_{n=1}^{N} \sum_{i=n-k+1}^{n} (y(t_i) - \alpha_n)^2. \tag{6}$$

In that paper we showed that the optimal control and splines converge under mild assumptions on the data. Mainly that the ϵ_{in}'s were symmetrically distributed and that first and second moments existed. The problem with this approach is that the linear systems that must be solved grow without bound creating insurmountable numerical difficulties.

In the next section we simplify the problem but the same basic questions remain.

3 A reduced problem

In this section we recast the problem as one with only discrete data. This simplification greatly reduces the complexity of the problem. We approximate the cost function $J_n(u)$ with a new cost function $J_n^j(u)$ by replacing the second integral with a finite sum.

A quadrature scheme is defined by of two data sets given by two real lower triangular matrices whose jth rows are given by

$$T_j = (r_{j1}, \cdots, r_{jj}, 0, 0, \cdots)$$

and

$$S_j = (\beta_{j1}, \cdots, \beta_{jj}, 0, 0, \cdots).$$

Let f be any continuous function and let

$$E_j(f) = \left| \int_0^T f(t)dt - \sum_{i=1}^{j} \beta_{ji} f(r_{ji}) \right|.$$

The quadrature scheme is convergent if for every $f \in C[0, T]$ we have the

$$\lim_{j \to \infty} E_j(f) = 0.$$

Such schemes abound, see for example [2].

Choose a particular convergent scheme. We now define a new cost function

$$J_n^j(u) = \int_0^T u(t)^2 dt + \lambda_n \sum_{i=1}^{j} \beta_{ij}(y(r_{ij}) - y_{n-1}(r_{ij}))^2 + \sum_{i=n-k+1}^{n} (y(t_i) - \alpha_i)^2 \quad (7)$$

in terms of the convergent quadrature scheme. We again rephrase the optimization problem as a minimum norm problem in Hilbert space. Here the past data is encoded in the spline function but we are only taking particular snapshots of the spline. The relationship between thee two problems clearly depends on the accuracy of the chosen quadrature scheme. However using the methods of [3] and [11] it is clear that as the solutions of J_n^j converge to those of J_n as j approaches infinity.

4 Solutions of the two problems

In this section we solve both the problem as formulated in Section 2 and discrete version of the problem as described in Section 3. we first consider the continuous case and reduce the solution to a system of integral equations.

4.1 The Continuous Case

We follow the development of [12]. First we describe the constrain variety. For the sake of simplicity we assume that the initial condition for the spline generator is 0. Let

$$\begin{aligned} V^k &= \{(u, y, \hat{y}(t_i)) : \ y(t) = \int_0^t ce^{A(t-s)} bu(s)ds, \ y(t_i) = \int_0^{t_n} \ell_i(s)u(s)ds\} \\ &\subset L_2[0, t_n] \times L_2[0, t_n] \times R^k \end{aligned}$$

Now we follow the standard procedure of constructing $V_0^{\perp}$, calculating the intersection

$$V_{x_0} \cap V_0 + p_0$$

and after some manipulation we have the following recursion for the optimal spline.

$$\begin{aligned} \lambda_k^{-1} y_{n+1} + L(t)(y_{n+1} - y_n) &+ H_k(I + G_k)^{-1} \hat{L}_k(y_n - y_{n+1}) \\ &= \lambda_k^{-1}(-H_k(I + G_k)^{-1} G_k \hat{\alpha} \\ &\quad -(I + G_k)^{-1} C_k + H_k \hat{\alpha}) \end{aligned}$$

4.2 The Discrete Case

In this case the Hilbert space is

$$L_2[0, T] \times R^{N+j}$$

withe the norm defined as

$$\|u, x)\|^2 = \int_o^T u^2(t)dt + x'Qx$$

where

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix}$$

and

$$Q_1 = \text{Diagonal}(\lambda_k\omega_{1j}, \cdots, \lambda_k\omega_{jj}).$$

Since there are two different time sequences we define

$$\wp_{ij}(s) = \begin{cases} ce^{A(r_{ij}-s)}b & r_{ij} - s \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The linear variety is

$$V_{x_0} = \{(u, (\rho, \gamma)) : \ \rho_{ij} = ce^{Ar_{ij}}x_0 + \int_0^T \wp_{ij}(s)u(s)ds, \ \ \gamma_i = ce^{At_i}x_0 + \int_0^T \ell_i(s)u(s)ds\}$$

The only complication that arises is we must make sure that the quadrature nodes and the time values for the data do not coincide. If they coincide the we loose independence of the basis functions. Other than that the procedure is exactly the same as developed in detail in [4].

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