Quantile correlations and quantile autoregressive modeling

Guodong Li, Yang Li and Chih-Ling Tsai

University of Hong Kong and University of California at Davis

December 27, 2012

Abstract

In this paper, we propose two important measures, quantile correlation (QCOR) and quantile partial correlation (QPCOR). We then apply them to quantile autoregressive (QAR) models, and introduce two valuable quantities, the quantile autocorrelation function (QACF) and the quantile partial autocorrelation function (QPACF). This allows us to extend the Box-Jenkins three stage procedure (model identification, model parameter estimation, and model diagnostic checking) from classical autoregressive models to quantile autoregressive models. Specifically, the QPACF of an observed time series can be employed to identify the autoregressive order, while the QACF of residuals obtained from the fitted model can be used to assess the model adequacy. We not only demonstrate the asymptotic properties of QCOR, QPCOR, QACF, and PQACF, but also show the large sample results of the QAR estimates and the quantile version of the Ljung-Box test. Moreover, we obtain the bootstrap approximations to the distributions of parameter estimator and proposed measures. Simulation studies indicate that the proposed methods perform well in finite samples, and an empirical example is presented to illustrate usefulness.

Keywords and phrases: Autocorrelation function; Bootstrap method; Box-Jenkins method; Quantile correlation; Quantile partial correlation; Quantile autoregressive model

1 Introduction

In the last decade, quantile regression has attracted considerable attention. There are two major reasons for such popularity. The first is that quantile regression estimation (Koenker and Bassett, 1978) can be robust to non-Gaussian or heavy-tailed data, and it includes the commonly used least absolute deviation (LAD) method as a special case. The second is that the quantile regression model allows practitioners to provide more easily interpretable regression estimates obtained via various quantiles $\tau \in [0, 1]$. More references about quantile regression estimation and interpretation can be found in the seminal book of Koenker (2005). Further extension of quantile regression to various model and data structures can be found in the literature, e.g., Machado and Silva (2005) for count data, Mu and He (2007) for power transformed data, Peng and Huang (2008) and Wang and Wang (2009) for survival analysis, He and Liang (2000) and Wei and Carroll (2009) for regression with measurement errors, Ando and Tsay (2011) for regression with augmented factors, and Kai et al. (2011) for semiparametric varying-coefficient partially linear models, among others.

In addition to the regression context, the quantile technique has been employed to the field of time series; see, for example, Koul and Saleh (1995) and Cai et al. (2012) for autoregressive (AR) models, Ling and McAleer (2004) for unstable AR models, and Xiao and Koenker (2009) for generalized autoregressive conditional heteroscedastic (GARCH) models. In particular, Koenker and Xiao (2006) established important statistical properties for quantile autoregressive (QAR) models, which expanded the classical AR model into a new era. In AR models, Box and Jenkins' (1970) three-stage procedure (i.e., model identification, model parameter estimation, and model diagnostic checking) has been commonly used for the last forty years. This motivates us to extend the classical Box-Jenkins' approach from AR to QAR models. In the classical AR model, it is known that model identification usually relies on the partial autocorrelation function (PACF) of the observed time series, while model diagnosis commonly depends on the autocorrelation function (ACF) of model residuals. Detailed illustrations of model identification and diagnosis can be found in Box et al. (2008). The aim of this paper is to introduce two novel measures to examine the linear and partially linear relationships between any two random variables for the given quantile $\tau \in [0, 1]$. We name them quantile correlation (QCOR) and quantile partial correlation (QPCOR). Based on these two measures, we propose the quantile partial autocorrelation function (QPACF) and the quantile autocorrelation function (QACF) to identify the order of the QAR model and to assess model adequacy, respectively. We also employ the bootstrap approach to study the performance of QPACF and QACF. It is noteworthy that the application of QCOR and QPCOR is not limited to QAR models. They can be used broadly as the classical correlation and partial correlation measures in various contexts.

The rest of this article is organized as follows. Section 2 introduces QCOR and QPCOR. Furthermore, the asymptotic properties of their sample estimators are established. Section 3 obtains the autoregressive parameter estimator and its asymptotic distribution. Then, QPACF and its large sample property for identifying the order of QAR models are demonstrated. Subsequently, QACF and its resulting test statistics, together with their asymptotic results, are provided to examine the model adequacy. In addition, bootstrap approximations to the distributions of parameter estimators, the QPACF measure, and the QACF measure are studied. Section 4 conducts simulation experiments to assess the finite sample performance of the proposed methods, and Section 5 presents an empirical example to demonstrate their usefulness. Finally, we conclude the article with a brief discussion in Section 6. All technical proofs of lemmas and theorems are relegated to the Appendix.

2 Correlations

2.1 Quantile correlation and quantile partial correlation

For random variables X and Y, let $Q_{\tau,Y}$ be the τth unconditional quantile of Y and $Q_{\tau,Y}(X)$ be the τth quantile of Y conditional on X. One can show that $Q_{\tau,Y}(X)$ is independent of X, i.e. $Q_{\tau,Y}(X) = Q_{\tau,Y}$ with probability one, if and only if the random variables $I(Y - Q_{\tau,Y} > 0)$ and X are independent, where $I(\cdot)$ is the indicator function. This fact has been used by He and Zhu (2003) and Mu and He (2007), and it also motivates us to define the quantile covariance given below. For $0 < \tau < 1$, define

$$\text{qcov}_\tau\{Y, X\} = \text{cov}\{I(Y - Q_{\tau,Y} > 0), X\} = E\{\psi_\tau(Y - Q_{\tau,Y})(X - EX)\},$$

where the function $\psi_\tau(w) = \tau - I(w < 0)$. Subsequently, the quantile correlation can be defined as follows,

$$\text{qcor}_\tau\{Y, X\} = \frac{\text{qcov}_\tau\{Y, X\}}{\sqrt{\text{var}\{\psi_\tau(Y - Q_{\tau,Y})\}\text{var}(X)}} = \frac{E\{\psi_\tau(Y - Q_{\tau,Y})(X - EX)\}}{\sqrt{(\tau - \tau^2)\sigma_X^2}}, \quad (2.1)$$

where $\sigma_X^2 = \text{var}(X)$.

In the simple linear regression with the quadratic loss function, there is a nice relationship between the slope and correlation. Hence, it is of interest to find a connection between the quantile slope and $\text{qcov}_\tau\{Y, X\}$. To this end, consider a simple quantile linear regression,

$$(a_0, b_0) = \underset{a,b}{\text{argmin}}\, E[\rho_\tau(Y - a - bX)],$$

in which one attempts to approximate $Q_{\tau,Y}(X)$ by a linear function $a_0 + b_0 X$ (see Koenker, 2005), where $\rho_\tau(w) = w[\tau - I(w < 0)]$. Then, we obtain the relationship between b_0 and $\text{qcor}_\tau\{Y, X\}$ given below.

Lemma 1. *Suppose that random variables X and Y have a joint density and $EX^2 < \infty$. Then the values of (a_0, b_0) are unique, and the quantity $b_0 = 0$ if and only if the quantile correlation $qcor_\tau\{Y, X\} = 0$.*

It is noteworthy that the proposed quantile covariance here does not enjoy the symmetry property of the classical covariance, i.e., $\text{qcov}_\tau(Y, X) \neq \text{qcov}_\tau(X, Y)$. This is because the first argument of the quantile covariance or the quantile correlation is related to the τth quantile, while the second argument is the same as that of the classical covariance. Accordingly, $\text{qcor}_\tau(Y, X) \neq \text{qcor}_\tau(X, Y)$.

Suppose that a quantile linear regression model has the response Y, a $q \times 1$ vector of covariate $\mathbf{Z}$, and an additional covariate X. In the classical regression model, one can construct the partial correlation to measure the linear relationship between variables Y and X after adjusting for (or controlling for) vector $\mathbf{Z}$ (e.g., see Chatterjee and Hadi, 2006). This motivates us to propose the quantile partial correlation function. To this end, let

$$(\alpha_1, \beta_1') = \underset{\alpha,\beta}{\text{argmin}}\, E(X - \alpha - \beta'\mathbf{Z})^2,$$

where $(\alpha, \beta')'$ is a vector of unknown parameters. Accordingly, $\alpha_1 + \beta_1'\mathbf{Z}$ is the linear effect of $\mathbf{Z}$ on X. Next, consider

$$(\alpha_2, \beta_2') = \underset{\alpha,\beta}{\operatorname{argmin}} E[\rho_\tau(Y - \alpha - \beta'\mathbf{Z})].$$

As a result, $\alpha_2+\beta_2'\mathbf{Z}$ is the linear effect of $\mathbf{Z}$ on the quantile Y (i.e., the linear approximation of $Q_{\tau,Y}(\mathbf{Z})$). It can also be shown that $E(X - \alpha_1 - \beta_1'\mathbf{Z}) = 0$, $E[\psi_\tau(Y - \alpha_2 - \beta_3'\mathbf{Z})] = 0$ and $E[\psi_\tau(Y - \alpha_2 - \beta_3'\mathbf{Z})\mathbf{Z}] = 0$ if the random vector $(X, Y, \mathbf{Z}')'$ satisfies the conditions stated in the forthcoming Lemma 2. Using these facts, we define the quantile partial correlation as follows,

$$\begin{aligned}
\text{qpcor}_\tau\{Y, X|\mathbf{Z}\} &= \frac{\text{cov}\{\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z}), X - \alpha_1 - \beta_1'\mathbf{Z}\}}{\sqrt{\text{var}\{\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})\}\text{var}\{X - \alpha_1 - \beta_1'\mathbf{Z}\}}} \\
&= \frac{E[\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})(X - \alpha_1 - \beta_1'\mathbf{Z})]}{\sqrt{(\tau - \tau^2)E(X - \alpha_1 - \beta_1'\mathbf{Z})^2}} \\
&= \frac{E[\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})X]}{\sqrt{(\tau - \tau^2)\sigma^2_{X|\mathbf{Z}}}}, \qquad (2.2)
\end{aligned}$$

where $\sigma^2_{X|\mathbf{Z}} = E(X - \alpha_1 - \beta_1'\mathbf{Z})^2$. This indicates that the covariate X has no additional linear contribution to the quantile response Y if $\alpha_2+\beta_2'\mathbf{Z} = \alpha_3+\beta_3'\mathbf{Z}+\gamma_3 X$ with probability one, where

$$(\alpha_3, \beta_3', \gamma_3) = \underset{\alpha,\beta,\gamma}{\operatorname{argmin}} E[\rho_\tau(Y - \alpha - \beta'\mathbf{Z} - \gamma X)].$$

This leads to the following lemma.

Lemma 2. *Suppose that the random vector $(X, Y, \mathbf{Z}')'$ has a joint density with $EX^2 < \infty$ and $E\|\mathbf{Z}\|^2 < \infty$, where $\| \cdot \|$ is the Euclidean norm. Then $(\alpha_3, \beta_3', \gamma_3) = (\alpha_2, \beta_2', 0)$ if and only if the quantile partial correlation qpcor$_\tau\{Y, X|\mathbf{Z}\} = 0$.*

Since the true qcor$_\tau$ and qpcor$_\tau$ are often unknown in practice, we introduce their sample versions given below.

2.2 Sample quantile correlation and sample quantile partial correlation

Suppose that the data $\{(Y_i, X_i, \mathbf{Z}_i')', i = 1, ..., n\}$ are identically and independently generated from a distribution of $(Y, X, \mathbf{Z}')'$. Let $\widehat{Q}_{\tau,Y} = \inf\{y : F_n(y) \geq \tau\}$ be the sample τth quantile of $Y_1, ..., Y_n$, where $F_n(y) = n^{-1} \sum_{i=1}^n I(Y_i \le y)$ is the empirical distribution function. Based on equation (2.1), the sample estimate of the quantile correlation $\text{qcor}_\tau\{Y, X\}$ is defined as

$$\widehat{\text{qcor}}_\tau\{Y, X\} = \frac{1}{\sqrt{(\tau - \tau^2)\widehat{\sigma}_X^2}} \cdot \frac{1}{n} \sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}), \tag{2.3}$$

where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$, and $\widehat{\sigma}_X^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$.

To study the asymptotic property of $\widehat{\text{qcor}}_\tau\{Y, X\}$, denote $f_Y(\cdot)$ and $f_{Y|X}(\cdot)$ as the density of Y and the conditional density of Y given X, respectively. In addition, let $\mu_X = E(X)$, $\mu_{X|Y} = E[f_{Y|X}(Q_{\tau,Y})X]/f_Y(Q_{\tau,Y})$, $\Sigma_{11} = E(X - \mu_X)^4 - \sigma_X^4$,

$$\Sigma_{12} = E[\psi_\tau(Y - Q_{\tau,Y})(X - \mu_{X|Y})]^2 - [\text{qcov}_\tau\{Y, X\}]^2,$$

$$\Sigma_{13} = E[\psi_\tau(Y - Q_{\tau,Y})(X - \mu_{X|Y})(X - \mu_X)^2] - \sigma_X^2 \cdot \text{qcov}_\tau\{Y, X\},$$

and

$$\Omega_1 = \frac{1}{\tau - \tau^2} \left[\frac{\Sigma_{11}(\text{qcov}_\tau\{Y, X\})^2}{4\sigma_X^6} - \frac{\Sigma_{13} \cdot \text{qcov}_\tau\{Y, X\}}{\sigma_X^4} + \frac{\Sigma_{12}}{\sigma_X^2} \right],$$

where σ_X^2 is defined as in the previous subsection. Then, we obtain the following result.

Theorem 1. *Suppose that $E(X)^4 < \infty$ and there exists a $\pi > 0$ such that the density $f_Y(\cdot)$ is continuous and the conditional density $f_{Y|X}(\cdot)$ is uniformly integrable on $[Q_{\tau,Y} - \pi, Q_{\tau,Y} + \pi]$. Then*

$$\sqrt{n} \left(\widehat{qcor}_\tau\{Y, X\} - qcor_\tau\{Y, X\} \right) \to_d N(0, \Omega_1).$$

To apply the above theorem, one needs to estimate the asymptotic variance Ω_1. To this end, we employ a nonparametric approach, such as the Nadaraya-Watson regression, to estimate the function $m(y) = E(X|Y = y)$, and denote the estimator by $\widehat{m}(y)$. We further assume that the random vector (X, Y) has a joint density, which leads to $\mu_{X|Y} = E(X|Y = Q_{\tau,Y})$. Accordingly, we obtain the estimate, $\widehat{\mu}_{X|Y} = \widehat{m}(\widehat{Q}_{\tau,Y})$, where $\widehat{Q}_{\tau,Y}$ is the τth sample quantile of $\{Y_1, ..., Y_n\}$. Finally, the rest of the quantities contained in Ω_1 (μ_X, σ_X^2, $\text{qcov}_\tau\{Y, X\}$, Σ_{11}, Σ_{12}, and Σ_{13}) can be, respectively, estimated by $\widehat{\mu}_X = \bar{X} = n^{-1} \sum_{i=1}^n X_i$, $\widehat{\sigma}_X^2 = n^{-1} \sum_{i=1}^n (X_i - \widehat{\mu}_X)^2$, $\widehat{\text{qcov}}_\tau\{Y, X\} = n^{-1} \sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i -$

$\bar{X})$, $\widehat{\Sigma}_{11} = n^{-1}\sum_{i=1}^n (X_i - \widehat{\mu}_X)^4 - \widehat{\sigma}_X^4$, $\widehat{\Sigma}_{12} = n^{-1}\sum_{i=1}^n [\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \widehat{\mu}_{X|Y})]^2 - [\widehat{\text{qcov}}_\tau\{Y,X\}]^2$, and $\widehat{\Sigma}_{13} = n^{-1}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \widehat{\mu}_{X|Y})(X_i - \widehat{\mu}_X)^2 - \widehat{\sigma}_X^2 \widehat{\text{qcov}}_\tau\{Y,X\}$. As a result, we obtain an estimate of Ω_1, and denote it by $\widehat{\Omega}_1$.

We next estimate the quantile partial correlation $\text{qpcor}_\tau\{Y,X\}$. Let

$$(\widehat{\alpha}_1, \widehat{\beta}_1') = \underset{\alpha,\beta}{\text{argmin}} \sum_{i=1}^n (X_i - \alpha - \beta'\mathbf{Z}_i)^2 \quad \text{and} \quad (\widehat{\alpha}_2, \widehat{\beta}_2') = \underset{\alpha,\beta}{\text{argmin}} \sum_{i=1}^n \rho_\tau(Y_i - \alpha - \beta'\mathbf{Z}_i).$$

Based on equation (2.2), the sample quantile partial correlation is defined as

$$\widehat{\text{qpcor}}_\tau\{Y, X|\mathbf{Z}\} = \frac{1}{\sqrt{(\tau - \tau^2)\widehat{\sigma}_{X|\mathbf{Z}}^2}} \cdot \frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{\alpha}_2 - \widehat{\beta}_2'\mathbf{Z}_i)X_i, \tag{2.4}$$

where $\widehat{\sigma}_{X|\mathbf{Z}}^2 = n^{-1}\sum_{i=1}^n (X_i - \widehat{\alpha}_1 - \widehat{\beta}_1'\mathbf{Z}_i)^2$.

To investigate the asymptotic property of $\widehat{\text{qpcor}}_\tau\{Y, X|\mathbf{Z}\}$, denote the conditional density of Y given $\mathbf{Z}$ and the conditional density of Y given $\mathbf{Z}$ and X by $f_{Y|\mathbf{Z}}(\cdot)$ and $f_{Y|\mathbf{Z},X}(\cdot)$, respectively. In addition, let $\theta_1 = (\alpha_1, \beta_1')'$, $\theta_2 = (\alpha_2, \beta_2')'$, $\mathbf{Z}^* = (1, \mathbf{Z}')'$, $\Sigma_{21} = E[f_{Y|\mathbf{Z},X}(\theta_2'\mathbf{Z}^*)X\mathbf{Z}^*]$, $\Sigma_{22} = E[f_{Y|\mathbf{Z}}(\theta_2'\mathbf{Z}^*)\mathbf{Z}^*\mathbf{Z}^{*\prime}]$, $\Sigma_{20} = \Sigma_{21}'\Sigma_{22}^{-1}$, $\Sigma_{23} = E(X - \theta_1'\mathbf{Z}^*)^4 - \sigma_{X|\mathbf{Z}}^4$,

$$\Sigma_{24} = E[\psi_\tau(Y - \theta_2\mathbf{Z}^*)(X - \Sigma_{20}\mathbf{Z}^*)]^2 - \{E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X]\}^2,$$

$$\Sigma_{25} = E[\psi_\tau(Y - \theta_2\mathbf{Z}^*)(X - \Sigma_{20}\mathbf{Z}^*)(X - \theta_1'\mathbf{Z}^*)^2] - \sigma_{X|\mathbf{Z}}^2 \cdot E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X],$$

and

$$\Omega_2 = \frac{1}{\tau - \tau^2}\left[\frac{\Sigma_{23}(E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X])^2}{4\sigma_{X|\mathbf{Z}}^6} - \frac{\Sigma_{25} \cdot E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X]}{\sigma_{X|\mathbf{Z}}^4} + \frac{\Sigma_{24}}{\sigma_{X|\mathbf{Z}}^2}\right],$$

where α_1, β_1, α_2, β_2 and $\sigma_{X|\mathbf{Z}}^2$ are defined as in the previous subsection. Then, we have the following result.

Theorem 2. *Suppose that $\Sigma_{21} < \infty$, $0 < \Sigma_{22} < \infty$, $EX^4 < \infty$, $E\|\mathbf{Z}\|^4 < \infty$, $E(\mathbf{Z}^*\mathbf{Z}^{*\prime}) > 0$, and there exists a $\pi > 0$ such that $f_{Y|\mathbf{Z}}(\theta_2'\mathbf{Z}^* + \cdot)$ and $f_{Y|\mathbf{Z},X}(\theta_2'\mathbf{Z}^* + \cdot)$ are uniformly integrable on $[-\pi, \pi]$. Then*

$$\sqrt{n}[\widehat{qpcor}_\tau\{Y, X|\mathbf{Z}\} - qpcor_\tau\{Y, X|\mathbf{Z}\}] \rightarrow_d N(0, \Omega_2).$$

To estimate the asymptotic variance Ω_2 given in Theorem 2, we consider $Y^* = Y - \theta_2'\mathbf{Z}^*$ and $\text{qcov}_\tau\{Y^*, X\} = E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X]$. In addition, assume that the random vector $(Y, X, \mathbf{Z}')'$ has a joint density. We then have that $\Sigma_{21} = E[f_{Y^*|\mathbf{Z},X}(0)X\mathbf{Z}^*] = f_{Y^*}(0) \cdot$

$E[X\mathbf{Z}^*|Y^*=0]$, $\Sigma_{22} = f_{Y^*}(0)\cdot E[\mathbf{Z}^*\mathbf{Z}^{*\prime}|Y^*=0]$, and $\Sigma_{20} = E[X\mathbf{Z}^{*\prime}|Y^*=0]\{E[\mathbf{Z}^*\mathbf{Z}^{*\prime}|Y^*=0]\}^{-1}$, where $f_{Y^*}(\cdot)$ is the density of Y^*. Applying the same nonparametric technique as that used for estimating $\mu_{X|Y}$ in Theorem 1, we can estimate each of the vector and matrix components in $\mathbf{m}_1(y) = E[X\mathbf{Z}^*|Y^*=y]$ and $\mathbf{m}_2(y) = E[\mathbf{Z}^*\mathbf{Z}^{*\prime}|Y^*=y]$, respectively, from the data $\{(Y_i^*, X_i, \mathbf{Z}_i') = (Y_i - \widehat{\theta}_2'\mathbf{Z}_i^*, X_i, \mathbf{Z}_i'), i=1,...,n\}$, where $\widehat{\theta}_2 = (\widehat{\alpha}_2, \widehat{\beta}_2')'$. Accordingly, $\widehat{\Sigma}_{20} = \widehat{\Sigma}_{21}'\widehat{\Sigma}_{22}^{-1} = \widehat{\mathbf{m}}_1'(0)[\widehat{\mathbf{m}}_2(0)]^{-1}$. Subsequently, the rest of the quantities involved in Ω_2, namely $\sigma^2_{X|\mathbf{Z}}$, $\text{qcov}_\tau\{Y^*, X\}$, Σ_{23}, Σ_{24}, and Σ_{25} can be, respectively, estimated by $\widehat{\sigma}^2_{X|\mathbf{Z}} = n^{-1}\sum_{i=1}^n (X_i - \widehat{\alpha}_1 - \widehat{\beta}_1'\mathbf{Z}_i)^2$, $\widehat{\text{qcov}}_\tau\{Y^*, X\} = n^{-1}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{\theta}_2'\mathbf{Z}_i^*)X_i$, $\widehat{\Sigma}_{23} = n^{-1}\sum_{i=1}^n (X_i - \widehat{\theta}_1'\mathbf{Z}_i^*)^4 - \widehat{\sigma}^4_{X|\mathbf{Z}}$, $\widehat{\Sigma}_{24} = n^{-1}\sum_{i=1}^n [\psi_\tau(Y_i - \widehat{\theta}_2\mathbf{Z}_i^*)(X_i - \widehat{\Sigma}_{20}\mathbf{Z}_i^*)]^2 - [\widehat{\text{qcov}}_\tau\{Y^*, X\}]^2$, and $\widehat{\Sigma}_{25} = n^{-1}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{\theta}_2\mathbf{Z}_i^*)(X_i - \widehat{\Sigma}_{20}\mathbf{Z}_i^*)(X_i - \widehat{\theta}_1'\mathbf{Z}_i^*)^2 - \widehat{\sigma}^2_{X|\mathbf{Z}}\cdot\widehat{\text{qcov}}_\tau\{Y^*, X\}$. Consequently, we obtain the estimate of Ω_2, and denote it by $\widehat{\Omega}_2$. We next apply the quantile correlation and quantile partial correlation to quantile autoregressive models.

3 Quantile autoregressive analysis

Suppose that $\{y_t\}$ is a strictly stationary and ergodic time series, and $\mathcal{F}_t$ is the σ-field generated by $\{y_t, y_{t-1}, ...\}$. We then follow Koenker and Xiao's (2006) approach and present QAR models; i.e., conditional on $\mathcal{F}_{t-1}$, the τth quantile of y_t has the form of

$$Q_\tau(y_t|\mathcal{F}_{t-1}) = \phi_0(\tau) + \phi_1(\tau)y_{t-1} + \cdots + \phi_p(\tau)y_{t-p} \text{ for } 0<\tau<1, \tag{3.1}$$

where the $\phi_i(\cdot)$s are unknown functions mapping from $[0,1] \to R$. Following the Box-Jenkins' three-stage procedure, we first introduce the QPACF of a time series to identify the order of a QAR model, then estimate model parameters, and finally use the QACF of residuals to assess the adequacy of the fitted model.

3.1 Model identification

For the positive integer k, let $\mathbf{z}_{t,k-1} = (y_{t-1}, ..., y_{t-k+1})'$, $(\alpha_1, \beta_1') = \text{argmin}_{\alpha,\beta} E(y_{t-k} - \alpha - \beta'\mathbf{z}_{t,k-1})^2$, and $(\alpha_2, \beta_2') = \text{argmin}_{\alpha,\beta} E[\rho_\tau(y_t - \alpha - \beta'\mathbf{z}_{t,k-1})]$, where the notations (α_1, β_1') and (α_2, β_2') are a slight abuse since they have been used to denote the regression parameters in Section 2. From equation (2.2), we obtain the quantile partial correlation between y_t

and y_{t-k} after adjusting the linear effect $\mathbf{z}_{t,k-1}$,

$$\phi_{kk,\tau} = \text{qpcor}_\tau\{y_t, y_{t-k}|\mathbf{z}_{t,k-1}\} = \frac{E[\psi_\tau(y_t - \alpha_2 - \beta_2'\mathbf{z}_{t,k-1})y_{t-k}]}{\sqrt{(\tau-\tau^2)E(y_{t-k} - \alpha_1 - \beta_1'\mathbf{z}_{t,k-1})^2}},$$

and it is independent of the time index t due to the strict stationarity of $\{y_t\}$. Analogously to the definition of the classical PACF (Fan and Yao, 2003, Chapter 2), we name $\phi_{kk,\tau}$ to be the QPACF of time series $\{y_t\}$. It is also noteworthy that $\phi_{11,\tau} = \text{qcor}_\tau\{y_t, y_{t-1}\}$. We next show the cut-off property of QPACF.

Lemma 3. *If $\phi_p(\tau) \neq 0$ with $p > 0$, $Ey_t^2 < \infty$ and $E[y_t - E(y_t|\mathcal{F}_{t-1})]^2 > 0$, then $\phi_{pp,\tau} \neq 0$, and $\phi_{kk,\tau} = 0$ for $k > p$.*

The above lemma indicates that the proposed QPACF plays the same role as that of PACF in the classical AR model identification.

In practice, one needs the sample estimate of QPACF. To this end, let

$$(\widetilde{\alpha}_1, \widetilde{\beta}_1') = \underset{\alpha,\beta}{\text{argmin}} \sum_{t=k+1}^{n} (y_{t-k} - \alpha - \beta'\mathbf{z}_{t,k-1})^2, \quad (\widetilde{\alpha}_2, \widetilde{\beta}_2') = \underset{\alpha,\beta}{\text{argmin}} \sum_{t=k+1}^{n} \rho_\tau(y_t - \alpha - \beta'\mathbf{z}_{t,k-1}),$$

and $\widetilde{\sigma}^2_{y|\mathbf{z}} = n^{-1}\sum_{t=k+1}^{n}(y_{t-k} - \widetilde{\alpha}_1 - \widetilde{\beta}_1'\mathbf{z}_{t,k-1})^2$. According to (2.4), we obtain the estimation for $\phi_{kk,\tau}$,

$$\widetilde{\phi}_{kk,\tau} = \frac{1}{\sqrt{(\tau-\tau^2)\widetilde{\sigma}^2_{y|\mathbf{z}}}} \cdot \frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\alpha}_2 - \widetilde{\beta}_2'\mathbf{z}_{t,k-1})y_{t-k},$$

and we term it the sample QPACF of the time series.

To study the asymptotic property of $\widetilde{\phi}_{kk,\tau}$, we introduce the following assumption, which is similar to Condition A.3 in Koenker and Xiao (2006).

Assumption 1. *$Ey_t^2 < \infty$, $E[y_t - E(y_t|\mathcal{F}_{t-1})]^2 > 0$, and there exists a $\pi > 0$ such that $f_{t-1}(\cdot)$ is uniformly integrable on $[-\pi, \pi]$.*

Furthermore, define the conditional quantile error

$$e_{t,\tau} = y_t - \phi_0(\tau) - \phi_1(\tau)y_{t-1} - \cdots - \phi_p(\tau)y_{t-p}. \tag{3.2}$$

By (3.1), the random variable $I(e_{t,\tau} > 0)$ is independent of y_{t-k} for any $k > 0$, and $(\alpha_2, \beta_2') = (\phi_0(\tau), \phi_1(\tau), ..., \phi_p(\tau), 0, ..., 0)$ for $k > p$. Let $f_{t-1}(\cdot)$ be the conditional density of $e_{t,\tau}$ on the σ-field $\mathcal{F}_{t-1}$, and $\mathbf{z}^*_{t,k-1} = (1, \mathbf{z}'_{t,k-1})' = (1, y_{t-1}, ..., y_{t-k+1})'$. Moreover, let $A_0 =$

$E[y_{t-k}\mathbf{z}^*_{t,k-1}]$, $A_1 = E[f_{t-1}(0)y_{t-k}\mathbf{z}^*_{t,k-1}]$, $\Sigma_{30} = E[\mathbf{z}^*_{t,k-1}\mathbf{z}^{*\prime}_{t,k-1}]$, $\Sigma_{31} = E[f_{t-1}(0)\mathbf{z}^*_{t,k-1}\mathbf{z}^{*\prime}_{t,k-1}]$, and

$$\Omega_3 = \frac{E(y_t^2) - 2A_1'\Sigma_{31}^{-1}A_0 + A_1'\Sigma_{31}^{-1}\Sigma_{30}\Sigma_{31}^{-1}A_1}{E(y_{t-k} - \alpha_1 - \beta_1'\mathbf{z}_{t,k-1})^2}.$$

Then, we obtain the asymptotic result given below.

Theorem 3. *For $k > p$, if $A_1 < \infty$, $0 < \Sigma_{31} < \infty$ and Assumption 1 is satisfied, then $\phi_{kk,\tau} = 0$ and*

$$\sqrt{n}\widetilde{\phi}_{kk,\tau} \to_d N(0, \Omega_3).$$

To estimate Ω_3 in the above theorem, we first apply the Hendricks and Koenker (1991) method to obtain the estimation of $f_{t-1}(0)$ given below.

$$\widetilde{f}_{t-1}(0) = \frac{2h}{\widetilde{Q}_{\tau+h}(y_t|\mathcal{F}_{t-1}) - \widetilde{Q}_{\tau-h}(y_t|\mathcal{F}_{t-1})},$$

where $\widetilde{Q}_\tau(y_t|\mathcal{F}_{t-1}) = \widetilde{\phi}_0(\tau) + \widetilde{\phi}_1(\tau)y_{t-1} + \cdots + \widetilde{\phi}_k(\tau)y_{t-k}$ is the estimated τth quantile of y_t and h is the bandwidth selected via appropriate methods (e.g., see Koenker and Xiao, 2006). Afterwards, we can use the sample averaging to approximate A_0, A_1, Σ_{30}, Σ_{31}, $E(y_t^2)$, and $E(y_{t-k} - \alpha_1 - \beta_1'\mathbf{z}_{t,k-1})^2$ by replacing their $f_{t-1}(\cdot)$, α_1, and β_1, respectively, with $\widetilde{f}_{t-1}(0)$, $\widetilde{\alpha}_1$ and $\widetilde{\beta}_1$. Accordingly, we obtain an estimate of Ω_3, and denote it $\widehat{\Omega}_3$. In sum, we are able to use the threshold values $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$ to check the significance of $\widetilde{\phi}_{kk,\tau}$.

To demonstrate how to use the above theorem to identify the order of a QAR model, we generate the observations $y_1, ..., y_{200}$ from $y_t = \Phi^{-1}(u_t) + a(u_t)y_{t-1}$, where Φ is the standard normal cumulative distribution function, $a(x) = \max\{0.8 - 1.6x, 0\}$, and $\{u_t\}$ is an *i.i.d* sequence with uniform distribution on $[0, 1]$. We attempt to fit the QAR model (3.1) with $\tau = 0.2$, 0.4, 0.6, and 0.8, respectively, to the observed data $\{y_t\}$. Figure 1 presents the sample QPACF $\widetilde{\phi}_{kk,\tau}$ for each τ with the reference lines $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$. We may conclude that the order p is 1 when $\tau = 0.2$ and 0.4, while p is 0 when $\tau = 0.6$ and 0.8.

3.2 Model estimation

After the order p of model (3.1) is correctly identified, we subsequently fit the selected model to data. Let $\boldsymbol{\phi} = (\phi_0, \phi_1, ..., \phi_p)'$ be an any parameter vector in model (3.1) and

$\boldsymbol{\phi}(\tau) = (\phi_0(\tau), \phi_1(\tau), ..., \phi_p(\tau))'$ be the true value of $\boldsymbol{\phi}$. It is noteworthy that $(\alpha_2, \beta_2')'$ defined in Subsection 3.1 is $\boldsymbol{\phi}(\tau)$ when $k = p$. Consider

$$\widetilde{\boldsymbol{\phi}}(\tau) = \underset{\boldsymbol{\phi}}{\text{argmin}} \sum_{t=p+1}^{n} \rho_\tau(y_t - \boldsymbol{\phi}'\mathbf{z}_{t,p}^*),$$

where $\mathbf{z}_{t,p}^* = (1, \mathbf{z}_{t,p}')' = (1, y_{t-1}, ..., y_{t-p})'$. In addition, let $\Sigma_{40} = E[\mathbf{z}_{t,p}^*\mathbf{z}_{t,p}^{*\prime}]$, $\Sigma_{41} = E[f_{t-1}(0)\mathbf{z}_{t,p}^*\mathbf{z}_{t,p}^{*\prime}]$, and $\Omega_4 = (\tau - \tau^2)\Sigma_{41}^{-1}\Sigma_{40}\Sigma_{41}^{-1}$. We then obtain the following asymptotic property of the estimated parameter vector.

Theorem 4. *If* $0 < \Sigma_{41} < \infty$ *and Assumption 1 is satisfied, then*

$$\sqrt{n}\{\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau)\} \to_d N(0, \Omega_4).$$

The above result is similar to that of Theorem 2 in Koenker and Xiao (2006), although we make different assumptions. The Ω_4 in the above theorem can be estimated by applying the same techniques used for the estimation of Ω_3.

3.3 Model diagnostic checking

For the errors $\{e_{t,\tau}\}$ defined in (3.2), we employ equation (2.1) and the fact that $Q_{\tau,e_{t,\tau}} = 0$, and obtain QACF between $\{e_{t,\tau}\}$ and $\{e_{t-k,\tau}\}$ as follows,

$$\rho_{k,\tau} = \frac{E\{\psi_\tau(e_{t,\tau})[e_{t-k,\tau} - E(e_{t,\tau})]\}}{\sqrt{(\tau - \tau^2)\sigma_e^2}},$$

where $\sigma_e^2 = \text{var}(e_{t,\tau})$. Suppose that the QAR model is correctly specified. We can show that $\rho_{k,\tau} = 0$ for $k > 0$. Hence, we are able to use $\rho_{k,\tau}$ to assess the model fit. In the sample version, we consider the residuals of the QAR model,

$$\widetilde{e}_{t,\tau} = y_t - \widetilde{\phi}_0(\tau) - \widetilde{\phi}_1(\tau)y_{t-1} - \cdots - \widetilde{\phi}_p(\tau)y_{t-p},$$

for $t = p+1, ..., n$, and $\widetilde{e}_{t,\tau} = 0$ for $t = 1, ..., p$. It can be verified that the τth empirical quantile of $\{\widetilde{e}_{t,\tau}\}$ is zero. Based on this fact and equation (2.3), we obtain the estimation of $\rho_{k,\tau}$,

$$r_{k,\tau} = \frac{1}{\sqrt{(\tau - \tau^2)\widetilde{\sigma}_e^2}} \cdot \frac{1}{n} \sum_{t=k+1}^{n} \psi_\tau(\widetilde{e}_{t,\tau})(\widetilde{e}_{t-k,\tau} - \widetilde{\mu}_e),$$

where k is a positive integer, $\widetilde{\mu}_e = n^{-1}\sum_{t=k+1}^{n} \widetilde{e}_{t,\tau}$ and $\widetilde{\sigma}_e^2 = n^{-1}\sum_{t=k+1}^{n}(\widetilde{e}_{t,\tau} - \widetilde{\mu}_e)^2$. We name $r_{k,\tau}$ the sample QACF of residuals.

Adapting the classical linear time series approach (Li, 2004), we examine the significance of $\{r_{k,\tau}\}$ individually and jointly. For the given positive integer K, let $\mathbf{e}_{t-1,K} = (e_{t-1,\tau}, ..., e_{t-K,\tau})'$, $\Sigma_{50} = E[\mathbf{e}_{t-1,K}\mathbf{z}^{*\prime}_{t,p}]$, $\Sigma_{51} = E[f_{t-1}(0)\mathbf{e}_{t-1,K}\mathbf{z}^{*\prime}_{t,p}]$, and

$$\Omega_5 = \frac{1}{\sigma_e^2}\{E(\mathbf{e}_{t-1,K}\mathbf{e}'_{t-1,K}) + \Sigma_{51}\Sigma_{41}^{-1}\Sigma_{40}\Sigma_{41}^{-1}\Sigma'_{51} - \Sigma_{51}\Sigma_{41}^{-1}\Sigma'_{50} - \Sigma_{50}\Sigma_{41}^{-1}\Sigma'_{51}\}.$$

Then, we obtain the asymptotic distribution of $R_\tau = (r_{1,\tau}, ..., r_{K,\tau})'$ given below.

Theorem 5. *Assume that* $0 < \Sigma_{41} < \infty$, $\Sigma_{51} < \infty$, *and Assumption 1 holds. We then have*

$$\sqrt{n}R_\tau \to_d N(0, \Omega_5).$$

Applying the same techniques as used in the estimate of Ω_3, we are able to estimate the asymptotic variance Ω_5 and denote it $\widehat{\Omega}_5$. In addition, let the k-th diagonal element of $\widehat{\Omega}_5$ be $\widehat{\Omega}_{5k}$. Then, one can employ $r_{k,\tau}/\sqrt{\widehat{\Omega}_{5k}}$ to examine the significance of the k-th lag in the residual series.

To check the significance of R_τ jointly, it is natural to consider the test statistic $R'_\tau\widehat{\Omega}_5^{-1}R_\tau$. However, $\widehat{\Omega}_5$ may not be invertible. Hence, we approximate Ω_5 by $I_K - \sigma_e^{-2}\Sigma_{50}\Sigma_{40}^{-1}\Sigma'_{50}$, which holds under the assumption that $\{e_{t,\tau}\}$ is an independent and identically distributed ($i.i.d.$) sequence. The resulting matrix is idempotent and has rank $K-p$. This allows us to obtain a Box-Pierce type test statistic (Box and Pierce, 1970),

$$Q_{BP}(K) = n\sum_{j=1}^{K} r_{j,\tau}^2,$$

which follows an approximately chi-squared distribution with $K - p$ degrees of freedom, χ^2_{K-p}. Accordingly, $Q_{BP}(K)$ can be used to test the significance of $\rho_{1,\tau}$ to $\rho_{K,\tau}$ jointly.

3.4 Bootstrap approximations

To conduct model identification, parameter estimation, and model diagnostic checking, we need to estimate the variances of Ω_3, Ω_4, and Ω_5, respectively. Since those quantities involve the nonparametric estimate of the density function $f_{t-1}(0)$, it is essential to employ the bootstrap approach to investigate the performance of the proposed three-stage procedure for QAR models. In the context of quantile regression models, several bootstrap methods have been proposed; see, e.g., He and Hu (2002) and Kocherginsky et al. (2005). It is noteworthy that $f_{t-1}(0)$ often depends on the past observations so that the above methods may not be directly applicable to QAR models. Hence, we consider a bootstrap approach from Rao and Zhao (1992) by introducing a series of random weights to the loss function, see also Jin et al. (2001), Feng and He (2011) and Li et al. (2012).

In the rest of this subsection, we propose the following three bootstrap algorithms to approximate the distributions of $\widetilde{\phi}_{kk,\tau}$ in Theorem 3, $\widetilde{\boldsymbol{\phi}}(\tau)$ in Theorem 4, and R_τ in Theorem 5, respectively, which are used for model identification, parameter estimation, and model diagnostic checking.

I. Bootstrap algorithm for identification

I1. Generate the *i.i.d.* random weights $\{\omega_t\}$ from a non-negative distribution with mean one and variance one.

I2. Obtain the weighted quantile estimator of (α_2, β_2'),

$$(\widetilde{\alpha}_2^*, \widetilde{\beta}_2^{*\prime}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{t=k+1}^{n} \omega_t \rho_\tau(y_t - \alpha - \beta' \mathbf{z}_{t,k-1}),$$

and the weighted QPACF

$$\widetilde{\phi}_{kk,\tau}^* = \frac{1}{\sqrt{(\tau-\tau^2)\widetilde{\sigma}_{y|\mathbf{z}}^2}} \cdot \frac{1}{n} \sum_{t=k+1}^{n} \omega_t \psi_\tau(y_t - \widetilde{\alpha}_2^* - \widetilde{\beta}_2^{*\prime} \mathbf{z}_{t,k-1}) y_{t-k}.$$

Then, calculate the quantity $d_1^I = \widetilde{\phi}_{kk,\tau}^* - \widetilde{\phi}_{kk,\tau}$.

I3. Repeat steps I1-I2 $B-1$ times and obtain $d_2^I, ..., d_B^I$.

I4. Employ the empirical distribution of $\{d_1^I, ..., d_B^I\}$ to approximate the distribution of $\widetilde{\phi}_{kk,\tau}$.

II. Bootstrap algorithm for estimation

E1. Do step I1.

E2. Obtain the weighted quantile estimator of $\boldsymbol{\phi}(\tau)$,

$$\widetilde{\boldsymbol{\phi}}^*(\tau) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{t=p+1}^{n} \omega_t \rho_\tau(y_t - \alpha - \beta' \mathbf{z}_{t,p}),$$

and then calculate the quantity $\mathbf{d}_1^E = \widetilde{\boldsymbol{\phi}}^*(\tau) - \widetilde{\boldsymbol{\phi}}(\tau)$.

E3. Repeat steps E1-E2 $B-1$ times, and then obtain quantities $\mathbf{d}_2^E, ..., \mathbf{d}_B^E$.

E4. Employ the empirical distribution of $\{\mathbf{d}_1^E, ..., \mathbf{d}_B^E\}$ to approximate the distribution of $\widetilde{\boldsymbol{\phi}}(\tau)$.

III. Bootstrap algorithm for diagnosis

D1. Do step I1.

D2. Obtain the weighted quantile estimator $\widetilde{\boldsymbol{\phi}}^*(\tau)$ as given in E2 and the weighted sample QACF

$$r_{k,\tau}^* = \frac{1}{\sqrt{(\tau-\tau^2)\widetilde{\sigma}_e^2}} \cdot \frac{1}{n} \sum_{t=\max\{k,p\}+1}^{n} \omega_t \psi_\tau(y_t - \widetilde{\boldsymbol{\phi}}^{*\prime}(\tau)\mathbf{z}_{t,p}^*)(y_{t-k} - \widetilde{\boldsymbol{\phi}}^{*\prime}(\tau)\mathbf{z}_{t-k,p}^*),$$

where $\mathbf{z}_{t,p}^* = (1, \mathbf{z}_{t,p}')'$. Calculate the quantity $\mathbf{d}_1^D = R_\tau^* - R_\tau$, where $R_\tau^* = (r_{1,\tau}^*, ..., r_{K,\tau}^*)'$.

D3. Repeat steps D1 and D2 $B-1$ times, and then obtain $\mathbf{d}_2^D, ..., \mathbf{d}_B^D$.

D4. Employ the empirical distribution of $\{\mathbf{d}_1^D, ..., \mathbf{d}_B^D\}$ to approximate the distribution of R_τ.

We next provide theoretical properties for the above three bootstrap procedures.

Theorem 6. *Under assumptions of Theorems 3-5, it holds that, conditional on $y_1, ..., y_n$,*

(a) $\sqrt{n}(\widetilde{\phi}_{kk,\tau}^* - \widetilde{\phi}_{kk,\tau}) \to_d N(0, \Omega_3)$,

(b) $\sqrt{n}\{\widetilde{\boldsymbol{\phi}}^*(\tau) - \widetilde{\boldsymbol{\phi}}(\tau)\} \to_d N(0, \Omega_4)$,

(c) $\sqrt{n}(R_\tau^* - R_\tau) \to_d N(0, \Omega_5)$

in probability, where Ω_3, Ω_4, and Ω_5 are defined as in Theorems 3-5, respectively.

The above theorem allows us to approximate the distributions of $\widetilde{\phi}_{kk,\tau}$, $\widetilde{\boldsymbol{\phi}}(\tau)$, and R_τ via their corresponding bootstrap procedures for the QAR analysis of model identification, parameter estimation, and model diagnostic checking.

4 Simulation studies

In this section, we study finite sample performance of the proposed measures and tests. Specifically, Section 4.1 is for the sample quantile correlation and the sample quantile partial correlation, and Sections 4.2 and 4.3 assess the performance of $\widetilde{\phi}_{kk,\tau}$, $\widetilde{\boldsymbol{\phi}}(\tau)$, R_τ, and $Q_{BP}(K)$ under the assumption of *i.i.d.* and non-*i.i.d.* conditional quantile errors, respectively. In all experiments, we conduct 1,000 realizations for each combination of sample sizes $n = 100$, 200, and 500 and quantiles, $\tau = 0.25$, 0.50, and 0.75. In addition, the number of bootstrapped samples for the non-*i.i.d.* case is set to $B = 1,000$, and the random weights $\{\omega_t\}$ follow the standard exponential distribution.

4.1 Performance of QCOR and QPCOR

We generate the *i.i.d.* samples $\{(X_i, Y_i, Z_i), i = 1, ..., n\}$ from the following multivariate normal distribution,

$$(X, Y, Z) \sim N\left\{\mathbf{0}, \begin{pmatrix} 1.0 & 0.5 & 0.5 \\ 0.5 & 1.0 & 0.5 \\ 0.5 & 0.5 & 1.0 \end{pmatrix}\right\}.$$

After algebraic simplification, we obtain that

$$\text{qcor}_\tau\{Y, X\} = 0.5\exp\{-0.5[\Phi^{-1}(\tau)]^2\}/\sqrt{(\tau - \tau^2)2\pi},$$

and $\text{qpcor}_\tau\{Y, X|Z\} = \text{qcor}_\tau\{Y, X\}/\sqrt{3}$, where $\Phi(\cdot)$ is the cumulative standard normal distribution. Tables 1 and 2 present the bias (BIAS) and estimated standard deviation (ESD), respectively, of the sample quantile correlations $\widehat{\text{qcor}}_\tau\{Y, X\}$ and the sample quantile partial correlations $\widehat{\text{qpcor}}_\tau\{Y, X|Z\}$.

To estimate the asymptotic variances Ω_1 and Ω_2, we mainly need to estimate the quantities $\mu_{X|Y}$ and Σ_{20}, addressed in Subsection 2.2. To this end, we employ the Nadaraya-Watson approach with the two bandwidth selection methods proposed by Bofinger (1975) and Hall and Sheather (1988), respectively, which are given below.

$$h_B = n^{-1/5}\left\{\frac{4.5\phi^4(\Phi^{-1}(\tau))}{[2(\Phi^{-1}(\tau))^2 + 1]^2}\right\}^{1/5} \quad \text{and} \quad h_{HS} = n^{-1/3}z_\alpha^{2/3}\left\{\frac{1.5\phi^2(\Phi^{-1}(\tau))}{2(\Phi^{-1}(\tau))^2 + 1}\right\}^{1/3},$$

where $\phi(\cdot)$ is the standard normal density function, $z_\alpha = \Phi^{-1}(1-\alpha/2)$, for the construction of $1-\alpha$ confidence intervals, and α is set to 0.05. Furthermore, we consider two more bandwidths, $0.6h_B$ and $3h_{HS}$, suggested by Koenker and Xiao (2006). In sum, we have four bandwidth choices. The resulting asymptotic variance estimates, $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$, yield their corresponding asymptotic standard deviations (ASDs) given in Tables 1 and 2. Both tables indicate that the ASDs are close to their corresponding ESDs even when $n = 100$, and they become smaller as the sample size gets larger. In addition, biases are close to zero, and decrease as the sample size increases. Moreover, all four bandwidths lead to similar results.

4.2 QAR analysis with *i.i.d.* conditional quantile errors

To study the performance of QPACF for identifying the model order in the first stage of QAR analysis, we generate the data from the following process,

$$y_t = 0.5y_{t-1} + e_t, \tag{4.1}$$

where $\{e_t\}$ is an $i.i.d$ sequence with standard normal distribution. Under the above setting, it can be easily shown that the conditional quantile errors, $e_{t,\tau}$, are $i.i.d$ and $\Omega_3 = 1$. We then employ the approach of Hendricks and Koenker (1991) with the four bandwidths used in the previous subsection to estimate the density function, $f_{t-1}(0)$. This allows us to further estimate the variance Ω_3 in Theorem 3 (see Subsection 3.1). Table 3 presents the bias and estimated standard deviation of $\widetilde{\phi}_{kk,\tau}$ at $k = 2$, 3, and 4. It shows that biases are small even when $n = 100$, and the ESDs are close to the ASDs as well as their theoretical value $1/\sqrt{n}$.

To investigate the finite-sample performance of model estimates in the second stage of QAR analysis, we next use the same data generated from (4.1), and then fit it with the QAR model (3.1) of order $p = 1$. In addition, we employ the same approach as that given in the above study to estimate $f_{t-1}(0)$. As a result, the variance matrix Ω_4 in Theorem 4 can be estimated (see Subsection 3.1). Table 4 presents the biases, estimated standard deviations, and asymptotic standard deviations of parameter estimates $\widetilde{\phi}_0(\tau)$ and $\widetilde{\phi}_1(\tau)$. It shows that biases are close to zero even when the sample size is as small as $n = 100$. In addition, the ESDs are close to the ASDs, and both of them decrease as the sample size increases. Moreover, there is no discernible difference among the four bandwidths.

To examine the finite-sample performance of the sample QACF of residuals individually in the third stage of QAR analysis, we subsequently consider the same simulation settings as those in the first stage of the simulation experiment. Table 5 presents the biases, estimated standard deviations, and asymptotic standard deviations of $r_{k,\tau}$ at $k = 2$, 4, and 6. Apparently, biases are small and the ASDs are close to their corresponding ESDs.

Finally, we study the approximate test statistic $Q_{BP}(K)$, which is also a part of the third stage of QAR analysis. To this end, we generate data from the following process,

$$y_t = 0.5y_{t-1} + \phi y_{t-2} + e_t,$$

where $\{e_t\}$ are *i.i.d.* standard normal random variables. In addition, $\phi = 0$ corresponds to the null hypothesis, while $\phi \neq 0$ is associated with the alternative hypothesis. Moreover, the nominal level is 5%. Table 6 reports sizes and powers of $Q_{BP}(K)$ with $K = 6$. It shows that $Q_{BP}(K)$ controls the size well, and its power increases quickly when the sample size or ϕ becomes larger.

4.3 QAR analysis with non-*i.i.d.* conditional quantile errors

To assess the performance of the proposed measures for the non-*i.i.d.* conditional quantile errors, we generate the data from the following process,

$$y_t = 0.3y_{t-1} + 0.3\nu_t I(\nu_t > \chi^2_{0.35})y_{t-2} + \nu_t, \tag{4.2}$$

where $\{\nu_t\}$ are *i.i.d.* chi-squared random variables with one degree of freedom, and χ^2_α is the α-th quantile of ν_t such that $P(\nu_t < \chi^2_\alpha) = \alpha$. It is noteworthy that $\{y_t\}$ is a nonnegative time series, $Q_\tau(y_t|\mathcal{F}_{t-1}) = \chi^2_\tau + 0.3y_{t-1}$ for $\tau \leq 0.35$, and $Q_\tau(y_t|\mathcal{F}_{t-1}) = \chi^2_\tau + 0.3y_{t-1} + 0.3\chi^2_\tau y_{t-2}$ for $\tau > 0.35$. In other words, the resulting series is QAR(1) when $\tau \leq 0.35$, while it is QAR(2) when $\tau > 0.35$. Accordingly, the conditional quantile errors, $e_{t,\tau} = y_t - Q_\tau(y_t|\mathcal{F}_{t-1})$, depend on y_{t-2}, which are not *i.i.d.* random variables.

To understand the performance of $\widetilde{\phi}_{kk,\tau}$ in the first stage of model identification, Table 7 reports the biases, estimated standard deviations, and asymptotic standard deviations of $\widetilde{\phi}_{kk,\tau}$ at $k = 2$, 3 and 4 for $\tau = 0.25$ and at $k = 3$ and 4 for $\tau = 0.5$ and 0.75, respectively. It is noteworthy that we employ both direct and bootstrap methods to calculate the asymptotic standard deviation (ASD). The simulation results indicate that biases are small, and the ESDs are close to the ASDs computed via the direct method with four different bandwidths. In addition, the bootstrap approach yields similar results, which indicate that the standard deviation of $\widetilde{\phi}_{kk,\tau}$ is not sensitive to our non-*i.i.d.* settings. Since bootstrap is an approximate method, it is not surprising that bootstrap leads to slightly larger values than those computed from the direct method. Moreover, all biases, estimated standard deviations, and asymptotic standard deviations decrease as the sample size increases, which is consistent with theoretical findings.

We next study the second and third stages of model estimation and diagnostics. According to model (4.2), we fit QAR(1) for $\tau = 0.25$ and QAR(2) for $\tau = 0.5$ and 0.75. Furthermore, the sample QACF of residuals are calculated at $K = 6$. Table 8 presents the biases, estimated standard deviations, and asymptotic standard deviations of parameter estimates $\widetilde{\phi}_0(\tau)$ and $\widetilde{\phi}_1(\tau)$, and Table 9 reports these statistics for $r_{k,\tau}$ at $k = 2$, 4 and 6. Both tables show qualitatively similar findings to those in Table 7. It is of interest to note that the bandwidth $3h_{HS}$ in Table 8 yields the largest and smallest ASDs at $\tau = 0.25$ and $\tau = 0.75$, respectively, in comparison with the other three bandwidths for the estimation of $\phi_k(\tau)$. This finding is sensible since a larger bandwidth tends to favor QAR(2) rather than QAR(1).

Finally, we examine the approximate test statistic $Q_{BP}(K)$. To this end, we generate data from the following process,

$$y_t = 0.3y_{t-1} + 0.3\nu_t I(\nu_t > \chi^2_{0.35})y_{t-2} + \phi y_{t-3} + \nu_t,$$

where ν_t are defined in (4.2). For simplicity, the QAR(2) model is employed for three quantiles. Note that $\phi = 0$ corresponds to the null hypothesis, while $\phi > 0$ is associated with the alternative hypothesis. The nominal level of the test is 5%. Table 6 reports sizes and powers of $Q_{BP}(K)$ with $K = 6$. It shows that $Q_{BP}(K)$ controls the size well when n is large, and its power increases when the sample size or ϕ becomes larger. In addition, the model fitting with $\tau = 0.25$ contains more observations than that with $\tau = 0.75$. As a result, $\tau = 0.25$ yields larger power than $\tau = 0.75$.

Consequently, the above Monte Carlo studies support theoretical findings of our proposed quantile measures and tests. Since all four bandwidths perform similarly, we adopt Koenker and Xiao's (2006) approach and use $0.6h_B$ in the next empirical example.

5 Nasdaq Composite

This example considers the log return (as a percentage) of the daily closing price on the Nasdaq Composite from January 1, 2002 to December 31, 2007. There are 1,235 observations in total, and Figure 2 depicts the time series plot and the classical sample ACF. It is not surprising to conclude that these returns (i.e., log returns) are uncorrelated and can be treated as an evidence in support of the fair market theory. However, Veronesi (1999) found that the stock markets under-react to good news in bad times and over-react to bad news in good times. Hence, Baur et al. (2012) proposed aligning a good (bad) state with upper (lower) quantiles by fitting their stock returns data with the QAR(1) type models. This motivates us to employ the general QAR model with our proposed methods to explore the dependence pattern of stock returns at a lower quantile ($\tau = 0.2$), the median ($\tau = 0.5$), and an upper quantile ($\tau = 0.8$).

We first fit the returns at the lower quantile ($\tau = 0.2$), and then present its sample QPACF in Panel A of Figure 3. It shows that lags 1, 2, and 13 are significant, which suggests QAR(13) could be considered for model fitting. We then refine the model via the backward variable selection procedure at the 5% significance level. The resulting model is

$$\widehat{Q}_{0.2}(y_t|\mathcal{F}_{t-1}) = -0.4114_{0.0184} + 0.1117_{0.0299}y_{t-1} + 0.0951_{0.0292}y_{t-2} + 0.0992_{0.0283}y_{t-13}, \quad (5.1)$$

where the subscripts of parameter estimates are their associated standard errors, and the bandwidth $0.6h_B$ is employed in this whole section. Accordingly, the above coefficients are all significant at the 5% significance level. In addition, the second graph in Panel A presents the sample QPACF of residuals, and no lags stand out. This, together with the p-value of $Q_{BP}(18)$ being 0.742, implies that this model is adequate.

We next consider the scenario with $\tau = 0.5$. The sample QPACF in Panel B indicates that all lags are insignificant. Hence, we fit the following model,

$$\widehat{Q}_{0.5}(y_t|\mathcal{F}_{t-1}) = 0.0036_{0.0142}. \quad (5.2)$$

The above coefficient is not only small, but also not significant. In addition, none of the lags in the sample QACF of residuals in Panel B show significance. Moreover, the p-value of $Q_{BP}(18)$ is 0.566. Consequently, the above model is appropriate.

Finally, we study the upper quantile scenario with $\tau = 0.8$. The sample QPACF in Panel C exhibits that lags 1, 2, 7, 10 and 15 are significant, and suggests QAR(15) could be considered for model fitting. After refining the model via the backward variable selection procedure, we obtain

$$\begin{aligned}\widehat{Q}_{0.8}(y_t|\mathcal{F}_{t-1}) = 0.3988_{0.0161} - 0.1076_{0.0167}y_{t-1} - 0.0825_{0.0167}y_{t-2} \\ - 0.0790_{0.0219}y_{t-10} - 0.0802_{0.0181}y_{t-15}, \qquad (5.3)\end{aligned}$$

where all coefficients are significant at the 5% significance level. In addition, the sample QACF of residuals in Panel C displays that all lags are insignificant. This, in conjunction with the p-value of $Q_{BP}(18)$ being 0.215, indicates that the above model fits the data reasonably well.

Based on the three fitted QAR models, (5.1), (5.2), and (5.3), we obtain the following conclusions. (i.) The lag coefficients at the lower quantile ($\tau = 0.2$) are all positive. This indicates that if the returns in past days have been positive (negative), then today's negative return is alleviated (even lower). It also implies that stock markets under-react to good news in bad times. (ii.) The lag coefficients at the upper quantile ($\tau = 0.8$) are all negative. This shows that if the returns in past days have been negative (positive), then today's positive return ($\tau = 0.8$) is even higher (dampened). As a result, stock markets over-react to bad news in good times. (iii.) The intercept at ($\tau = 0.5$) has a small value and is insignificant at the 5% significance level. Thus, the conditional median of returns is almost zero as we expected. In addition, equation (5.2) indicates that today's return is not affected by the returns of recent past days. Although we only report the results of the lower and higher quantiles at $\tau = 0.2$ and $\tau = 0.8$, our studies yield the same conclusions across various lower and upper quantiles. Moreover, Figure 4 depicts the bootstrap results, which are consistent with those findings in Figure 3. In sum, our proposed methods support Veronesi's (1999) equilibrium explanation for stock market reactions.

6 Discussion

In quantile regression models, we propose the quantile correlation and quantile partial correlation. Then, we apply them to quantile autoregressive models, which yields the quantile autocorrelation and quantile partial autocorrelation. In practice, the response time series may depend on exogenous variables. Hence, it is of interest to extend those correlation measures to the quantile autoregressive model with the exogenous variables given below.

$$Q_\tau(y_t|\mathcal{F}_{t-1}) = \phi_0(\tau) + \sum_{i=1}^{p} \phi_i(\tau) y_{t-i} + \sum_{j=1}^{q} \beta_j'(\tau)\mathbf{x}_{t-j}, \text{ for } 0 < \tau < 1,$$

where $\mathbf{x}_t$ is a vector of time series, and $\phi_i(\tau)$ and $\beta_j(\tau)$ are functions $[0,1] \to R$, see Galvao et al. (2012). In addition, the application of the proposed correlations to the quantile regression model with autoregressive errors is worth further investigation. Clearly, the contribution of the proposed measures is not limited to those two models. For example, variable screening and selection (e.g., Fan and Lv 2008; Wang 2009) in quantile regressions are other important topics for future research. In sum, this paper introduces valuable measures to broaden and facilitate the use of quantile models.

Appendix: technical proofs

Proof of Lemma 1. For $a, b \in R$, denote the function $h(a,b) = E[\rho_\tau(Y^*)]$, where $Y^* = Y - a - bX$. We first show that $h(a,b)$ is a continuously differentiable function and has derivatives,

$$\frac{\partial h(a,b)}{\partial a} = -E[\psi_\tau(Y^*)] = P(Y^* < 0) - \tau \quad \text{and} \quad \frac{\partial h(a,b)}{\partial b} = -E[\psi_\tau(Y^*)X].$$

For $u \neq 0$,

$$\begin{aligned}\rho_\tau(u-v) - \rho_\tau(u) &= -v\psi_\tau(u) + \int_0^v [I(u \leq s) - I(u < 0)]ds \\ &= -v\psi_\tau(u) + (u-v)[I(0 > u > v) - I(0 < u < v)], \qquad \text{(A.1)}\end{aligned}$$

see Koenker and Xiao (2006). This, together with Hölder's inequality and the fact that $|Y^*|/|X|$ is a continuous random variable, leads to

$$\begin{aligned}
&|\frac{1}{c}[h(a,b+c)-h(a,b)]+E[\psi_\tau(Y^*)X]| \\
&\quad=|\frac{1}{c}E[\rho_\tau(Y^*-cX)-\rho_\tau(Y^*)]+E[\psi_\tau(Y^*)X]| \\
&\quad=|\frac{1}{c}E\{(Y^*-cX)[I(0>Y^*>cX)-I(0<Y^*<cX)]\}| \\
&\quad\le E[|X|I(|Y^*|<|c|\cdot|X|)]\le(EX^2)^{1/2}[P(|Y^*|/|X|<|c|)]^{1/2},
\end{aligned}$$

which tends to zero as $c\to 0$. Accordingly, $\partial h(a,b)/\partial b$ is obtained. Analogously, we have $\partial h(a,b)/\partial a$. By Hölder's inequality, we can further prove the continuity of $\partial h(a,b)/\partial b$. Moreover, the continuity of both X and Y implies that $\partial h(a,b)/\partial a$ is a continuous function. It is noteworthy that $h(a,b)$ is a convex function with $\lim_{a^2+b^2\to\infty}h(a,b)=+\infty$. This, in conjunction with the above results, demonstrates that the values of a_0 and b_0 satisfy

$$E[\psi_\tau(Y-a_0-b_0X)]=0 \quad \text{and} \quad E[\psi_\tau(Y-a_0-b_0X)X]=0. \tag{A.2}$$

We next show the uniqueness of (a_0,b_0). Suppose that there is another pair of values (a_1,b_1) such that $h(a_1,b_1)=h(a_0,b_0)=\text{argmin}_{a,b}E[\rho_\tau(Y-a-bX)]$. Let $Y_0=Y-a_0-b_0X$ and $\xi=(a_1-a_0)+(b_1-b_0)X$. Then, by (A.1) and (A.2),

$$\begin{aligned}
0&=h(a_1,b_1)-h(a_0,b_0)=E[\rho_\tau(Y_0-\xi)-\rho_\tau(Y_0)] \\
&=-E[\xi\psi_\tau(Y_0)]+E[(Y_0-\xi)I(0>Y_0>\xi)]+E[(\xi-Y_0)I(0<Y_0<\xi)] \\
&=E[(Y_0-\xi)I(0>Y_0>\xi)]+E[(\xi-Y_0)I(0<Y_0<\xi)].
\end{aligned}$$

Note that both $(Y_0-\xi)I(0>Y_0>\xi)$ and $(\xi-Y_0)I(0<Y_0<\xi)$ are nonnegative random variables, and $Y_0-\xi$ is a continuous random variable. Thus, with probability one, $I(0>Y_0>\xi)=I(0<Y_0<\xi)=0$, which implies $(a_1,b_1)=(a_0,b_0)$.

Finally, if $b_0=0$, then (A.2) leads to $a_0=Q_{\tau,Y}$ and $\text{qcov}_\tau\{Y,X\}=E[\psi_\tau(Y-a_0-b_0X)X]=0$. On the other hand, if $\text{qcov}_\tau\{Y,X\}=0$, then equation (A.2) with $(a_0,b_0)=(Q_{\tau,Y},0)$ holds. By the uniqueness property, $b_0=0$, which completes the proof.

Proof of Lemma 2. Let $Y^*=Y-\alpha_2-\beta_2'\mathbf{Z}$ and

$$(\alpha_4,\beta_4',\gamma_4)=\underset{\alpha,\beta,\gamma}{\text{argmin}}\,E[\rho_\tau(Y^*-\alpha-\beta'\mathbf{Z}-\gamma X)].$$

Since the random vector $(X, Y^*, \mathbf{Z}')'$ has a joint density, we apply similar techniques to those in the proof of Lemma 1 to show that

$$E[\psi_\tau(Y^*)] = 0, \quad E[\psi_\tau(Y^*)\mathbf{Z}] = \mathbf{0}, \tag{A.3}$$

and the values of α_4, β_4' and γ_4 are unique and satisfy

$$E[\psi_\tau(Y^* - \alpha_4 - \beta_4'\mathbf{Z} - \gamma_4 X)(1, \mathbf{Z}', X)'] = \mathbf{0}, \tag{A.4}$$

where $\mathbf{0}$ is a $(q+2) \times 1$ zero vector.

From (A.4), if $(\alpha_4, \beta_4', \gamma_4)' = \mathbf{0}$, then $\text{qcov}_\tau\{Y^*, X\} = E[\psi_\tau(Y^*)X] = 0$. On the other hand, $\text{qcov}_\tau\{Y^*, X\} = 0$, together with (A.3), implies that equation (A.4) with $(\alpha_4, \beta_4', \gamma_4)' = \mathbf{0}$ holds. Accordingly, we have shown that $\text{qcov}_\tau\{Y^*, X\} = 0$ if and only if $(\alpha_4, \beta_4', \gamma_4)' = \mathbf{0}$. Based on the definitions of (α_2, β_2') and $(\alpha_3, \beta_3', \gamma_3)$ in Subsection 2.1, we further have that $\alpha_4 = \alpha_3 - \alpha_2$, $\beta_4 = \beta_3 - \beta_2$, and $\gamma_4 = \gamma_3$. Finally, using the fact that $\text{qpcor}_\tau\{Y, X|\mathbf{Z}\} = \text{qcov}_\tau\{Y^*, X\}/\sqrt{(\tau - \tau^2)\sigma^2_{X|\mathbf{Z}}}$ completes the proof.

Proof of Lemma 3. For $k = p$, let

$$(\alpha_3, \beta_3', \gamma_3) = \underset{\alpha,\beta,\gamma}{\text{argmin}}\, E[\rho_\tau(y_t - \alpha - \beta'\mathbf{z}_{t,p-1} - \gamma y_{t-p})].$$

It is noteworthy that $(\alpha_3, \beta_3', \gamma_3) = (\phi_0(\tau), \phi_1(\tau), ..., \phi_p(\tau))$. Since $\phi_p(\tau) \neq 0$, we apply Lemma 2 and are able to show that $\phi_{pp,\tau} \neq 0$.

Let $e_{t,\tau} = y_t - \phi_0(\tau) - \phi_1(\tau)y_{t-1} - \cdots - \phi_p(\tau)y_{t-p}$. By (3.1), $I(e_{t,\tau} > 0)$ is independent of y_{t-k} for any $k > 0$. In addition, $(\alpha_2, \beta_2') = (\phi_0(\tau), \phi_1(\tau), ..., \phi_p(\tau), \mathbf{0}')$ for $k > p$, where $\mathbf{0}$ is $(k-p) \times 1$ vector. Hence, $\phi_{kk,\tau} = 0$ for $k > p$.

Proof of Theorem 1. For $u \neq 0$, we have that

$$I(u - v < 0) - I(u < 0) = I(v > u > 0) - I(v < u < 0).$$

Using this result, we then obtain

$$\frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}) = \frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i - Q_{\tau,Y})X_i + \frac{1}{n}A_n - \bar{X}\cdot\frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y}), \tag{A.5}$$

where $A_n = \sum_{i=1}^n g_\tau(Y_i, Q_{\tau,Y}, \widehat{Q}_{\tau,Y})X_i$ and

$$\begin{aligned} & g_\tau(Y_i, Q_{\tau,Y}, \widehat{Q}_{\tau,Y}) \\ &= \psi_\tau(Y_i - \widehat{Q}_{\tau,Y}) - \psi_\tau(Y_i - Q_{\tau,Y}) = -[I(Y_i < \widehat{Q}_{\tau,Y}) - I(Y_i < Q_{\tau,Y})] \\ &= I(\widehat{Q}_{\tau,Y} - Q_{\tau,Y} < Y_i - Q_{\tau,Y} < 0) - I(\widehat{Q}_{\tau,Y} - Q_{\tau,Y} > Y_i - Q_{\tau,Y} > 0). \end{aligned}$$

It can be shown that

$$|\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i-\widehat{Q}_{\tau,Y})| = |\tau - \frac{1}{n}\sum_{i=1}^{n} I(Y_i-\widehat{Q}_{\tau,Y})| = |\tau - \frac{[n\tau]}{n}| \leq \frac{1}{n}.$$

This, together with the law of large numbers, implies the last term of (A.5) satisfying

$$\bar{X}\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i-\widehat{Q}_{\tau,Y}) = O_p(n^{-1}). \quad (A.6)$$

We next consider the second term on the right-hand side of (A.5). For any $v \in R$, denote

$$\xi_n(v) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y}+n^{-1/2}v) - E[g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y}+n^{-1/2}v)|X_i]\}X_i,$$

where

$$E[g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y}+n^{-1/2}v)|X_i] = -\int_{Q_{\tau,Y}}^{Q_{\tau,Y}+n^{-1/2}v} f_{Y_i|X_i}(y)dy$$

and $f_{Y_i|X_i}(\cdot)$ is the conditional density of Y_i given X_i. Then, by Hölder's inequality, we have that

$$\begin{aligned} E[\xi_n(v)]^2 &= E[g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y}+n^{-1/2}v)X_i]^2 \\ &\leq [P(|Y_i - Q_{\tau,Y}| < n^{-1/2}v)]^{1/2}[EX_i^4]^{1/2} = o(1). \end{aligned} \quad (A.7)$$

After algebraic simplification, we further obtain

$$\begin{aligned} &\sup_{|v_1-v|<\delta}|\xi_n(v_1)-\xi_n(v)| \\ &\quad\leq \sup_{|v_1-v|<\delta}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}|\{g_\tau(v_1)-g_\tau(v)\}X_i| + E[|\{g_\tau(v_1)-g_\tau(v)\}X_i||X_i] \\ &\quad= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}|\{g_\tau(v_1^*)-g_\tau(v)\}X_i| + E[|\{g_\tau(v_1^*)-g_\tau(v)\}X_i||X_i], \end{aligned}$$

where v_1^* takes the value of $v+\delta$ or $v-\delta$. Hence,

$$\begin{aligned} &E\sup_{|v_1-v|<\delta}|\xi_n(v_1)-\xi_n(v)| \\ &\leq 2\sqrt{n}E|\{g_\tau(v_1^*)-g_\tau(v)\}X_i| \\ &= 2\sqrt{n}E\left|\int_{Q_{\tau,Y}+n^{-1/2}v}^{Q_{\tau,Y}+n^{-1/2}v_1^*} f_{Y_i|X_i}(y)dyX_i\right| \\ &\leq \delta\cdot 2E[\sup_{|y|\leq\pi} f_{Y_i|X_i}(Q_{\tau,Y}+y)|X_i|], \end{aligned} \quad (A.8)$$

where $|n^{-1/2}v| < \pi$ and $|n^{-1/2}v_1^*| < \pi$ when n is large. Both (A.7) and (A.8), in conjunction with the theorem's assumptions and the finite converging theorem, imply that $E\sup_{|v|\le M}|\xi_n(v)| = o(1)$ for any $M > 0$. In addition, applying the theorem in Section 2.5.1 of Serfling (1980), we have

$$\sqrt{n}(\widehat{Q}_{\tau,Y} - Q_{\tau,Y}) = f_Y^{-1}(Q_{\tau,Y}) \cdot \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_\tau(Y_i - Q_{\tau,Y}) + o_p(1) = O_p(1).$$

Accordingly,

$$\begin{aligned}
\frac{1}{\sqrt{n}}A_n &= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{Q_{\tau,Y}}^{Q_{\tau,Y}+(\widehat{Q}_{\tau,Y}-Q_{\tau,Y})} f_{Y_i|X_i}(y)dyX_i + o_p(1) \\
&= -(\widehat{Q}_{\tau,Y} - Q_{\tau,Y})\frac{1}{\sqrt{n}}\sum_{i=1}^{n} f_{Y_i|X_i}(Q_{\tau,Y})X_i + o_p(1) \\
&= -\frac{E[f_{Y_i|X_i}(Q_{\tau,Y})X_i]}{f_Y(Q_{\tau,Y})} \cdot \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_\tau(Y_i - Q_{\tau,Y}) + o_p(1). \qquad (A.9)
\end{aligned}$$

Subsequently, using (A.5), (A.6), and (A.9), we obtain that

$$\begin{aligned}
&\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}) - \text{qcov}_\tau\{Y,X\}\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}) - \text{qcov}_\tau\{Y,X\}] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\psi_\tau(Y_i - Q_{\tau,Y})(X_i - \mu_{X|Y}) - \text{qcov}_\tau\{Y,X\}] + o_p(1), \qquad (A.10)
\end{aligned}$$

where $\mu_{X|Y}$ is defined in Subsection 2.2. Since

$$\sqrt{n}(\bar{X} - \mu_X)^2 = \frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu_X)\right]^2 = O_p(n^{-1/2}),$$

we further have that

$$\sqrt{n}(\widehat{\sigma}_X^2 - \sigma_X^2) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[(X_i - \mu_X)^2 - \sigma_X^2] + o_p(1). \qquad (A.11)$$

Moreover, (A.10), (A.11), the central limit theorem, and the Cramer-Wold device, lead to

$$\sqrt{n}\begin{pmatrix} \widehat{\sigma}_X^2 - \sigma_X^2 \\ n^{-1}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}) - \text{qcov}_\tau\{Y,X\} \end{pmatrix} \to_d N(0,\Sigma),$$

where

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{13} & \Sigma_{12} \end{pmatrix},$$

and Σ_{11}, Σ_{12}, and Σ_{13} are defined in Subsection 2.2. Finally, following the Delta method (van der Vaart, 1998, Chapter 3), we complete the proof.

Proof of Theorem 2. We first consider the term $\widehat{\sigma}^2_{X|\mathbf{Z}}$ in $\widehat{\text{qpcor}}_\tau\{Y, X|\mathbf{Z}\}$. Let $\mathbf{Z}_i^* = (1, \mathbf{Z}_i')'$, $X_i^* = X_i - \alpha_1 - \beta_1'\mathbf{Z}_i$, $\theta_1 = (\alpha_1, \beta_1')'$ and $\widehat{\theta}_1 = (\widehat{\alpha}_1, \widehat{\beta}_1')'$, where (α_1, β_1') and $(\widehat{\alpha}_1, \widehat{\beta}_1')$ are defined in Subsections 2.1 and 2.2, respectively. By the assumptions of this theorem, we have that $\theta_1 = [E(\mathbf{Z}_i^*\mathbf{Z}_i^{*\prime})]^{-1}E(\mathbf{Z}_i^*X_i)$, $E(\mathbf{Z}_i^*X_i^*) = \mathbf{0}$, and

$$\sqrt{n}(\widehat{\theta}_1 - \theta_1) = \left(\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^*\mathbf{Z}_i^{*\prime}\right)^{-1} \cdot \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbf{Z}_i^*X_i^* = O_p(1).$$

According the law of large numbers, we then have that

$$\begin{aligned}\widehat{\sigma}^2_{X|\mathbf{Z}} &= \frac{1}{n}\sum_{i=1}^n (X_i - \widehat{\theta}_1'\mathbf{Z}_i^*)^2 \\ &= \frac{1}{n}\sum_{i=1}^n (X_i - \theta_1'\mathbf{Z}_i^*)^2 + (\widehat{\theta}_1 - \theta_1)'\left(\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^*\mathbf{Z}_i^{*\prime}\right)(\widehat{\theta}_1 - \theta_1) \\ &\qquad - 2(\widehat{\theta}_1 - \theta_1)'\left(\frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^*X_i^*\right) \\ &= \frac{1}{n}\sum_{i=1}^n (X_i - \theta_1'\mathbf{Z}_i^*)^2 + o_p(n^{-1/2}). \end{aligned} \tag{A.12}$$

We next consider the numerator in $\widehat{\text{qpcor}}_\tau\{Y, X|\mathbf{Z}\}$. For the sake of simplicity, let $Y_i^* = Y_i - \alpha_2 - \beta_2'\mathbf{Z}_i = Y_i - \theta_2\mathbf{Z}_i^*$, $\theta_2 = (\alpha_2, \beta_2')'$, and $\widehat{\theta}_2 = (\widehat{\alpha}_2, \widehat{\beta}_2')'$, where Y_i^* is defined in the proof of Lemma 2, and (α_2, β_2') and $(\widehat{\alpha}_2, \widehat{\beta}_2')$ are defined in Subsections 2.1 and 2.2, respectively. Under the theorem's assumptions, we employ similar techniques to those used in the proof of Lemma 1 and given in Koenker (2005) to show that there exists a unique θ_2 such that $E[\psi_\tau(Y_i^*)\mathbf{Z}_i^*] = \mathbf{0}$ and

$$\sqrt{n}(\widehat{\theta}_2 - \theta_2) = \{E[f_{Y_i|\mathbf{Z}_i}(\theta_2'\mathbf{Z}_i^*)\mathbf{Z}_i^*\mathbf{Z}_i^{*\prime}]\}^{-1} \cdot \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_\tau(Y_i^*)\mathbf{Z}_i^* + o_p(1). \tag{A.13}$$

Using a similar method to that for obtaining (A.5), we have that

$$\frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{\theta}_2\mathbf{Z}_i^*)X_i = \frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i^*)X_i + \frac{1}{n}\sum_{i=1}^n g_\tau(Y_i, \mathbf{Z}_i, \theta_2, \widehat{\theta}_2)X_i, \tag{A.14}$$

where

$$\begin{aligned} g_\tau(Y_i, \mathbf{Z}_i, \theta_2, \widehat{\theta}_2) &= \psi_\tau(Y_i - \widehat{\theta}_2\mathbf{Z}_i^*) - \psi_\tau(Y_i^*) = -[I(Y_i < \widehat{\theta}_2\mathbf{Z}_i^*) - I(Y_i < \theta_2\mathbf{Z}_i^*)] \\ &= I[(\widehat{\theta}_2 - \theta_2)'\mathbf{Z}_i^* < Y_i^* < 0] - I[(\widehat{\theta}_2 - \theta_2)'\mathbf{Z}_i^* > Y_i^* > 0]. \end{aligned}$$

For any $\mathbf{v} \in R^{q+1}$, let

$$\xi_n(\mathbf{v}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [g_\tau(Y_i, \mathbf{Z}_i, \theta_2, \theta_2 + n^{-1/2}\mathbf{v})X_i + \int_{\theta_2'\mathbf{Z}_i^*}^{\theta_2'\mathbf{Z}_i^* + n^{-1/2}\mathbf{v}'\mathbf{Z}_i^*} f_{Y_i|\mathbf{Z}_i, X_i}(y)dyX_i].$$

Applying similar techniques to those for obtaining (A.7) and (A.8), we can demonstrate that

$$\begin{aligned} E[\xi_n(\mathbf{v})]^2 &= E[g_\tau(Y_i, \mathbf{Z}_i, \theta_2, \theta_2 + n^{-1/2}\mathbf{v})X_i]^2 \\ &\leq \{P(|Y_i^*| \leq n^{-1/2}|\mathbf{v}'\mathbf{Z}_i^*|)\}^{1/2} \cdot (EX_i^4)^{1/2} = o(1) \end{aligned}$$

and, for any $\delta > 0$ and $\mathbf{v}_1 \in R^{p+1}$,

$$E \sup_{\|\mathbf{v}_1 - \mathbf{v}\| \leq \delta} |\xi_n(\mathbf{v}_1) - \xi_n(\mathbf{v})| \leq \delta \cdot 2E[\sup_{|y| \leq \pi} f_{Y_i|\mathbf{Z}_i, X_i}(\theta_2'\mathbf{Z}_i^* + y)|X_i|].$$

This implies that $E \sup_{\|\mathbf{v}\| \leq M} |\xi_n(\mathbf{v})| = o(1)$ for any $M > 0$. Note that, by (A.13), $\sqrt{n}(\widehat{\theta}_2 - \theta_2) = O_p(1)$. As a result,

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_\tau(Y_i, \mathbf{Z}_i, \theta_2, \widehat{\theta}_2)X_i \\ &\quad = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{\theta_2'\mathbf{Z}_i^*}^{\theta_2'\mathbf{Z}_i^* + (\widehat{\theta}_2 - \theta_2)'\mathbf{Z}_i^*} f_{Y_i|\mathbf{Z}_i, X_i}(y)dyX_i + o_p(1) \\ &\quad = -(\widehat{\theta}_2 - \theta_2)' \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_{Y_i|\mathbf{Z}_i, X_i}(\theta_2'\mathbf{Z}_i^*)X_i\mathbf{Z}_i^* + o_p(1) \\ &\quad = -\Sigma_{21}'\Sigma_{22}^{-1} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_\tau(Y_i^*)\mathbf{Z}_i^* + o_p(1), \end{aligned}$$

where $\Sigma_{21} = E\{f_{Y_i|\mathbf{Z}_i, X_i}(\theta_2'\mathbf{Z}_i^*)X_i\mathbf{Z}_i^*\}$ and $\Sigma_{22} = E[f_{Y_i|\mathbf{Z}_i}(\theta_2'\mathbf{Z}_i^*)\mathbf{Z}_i^*\mathbf{Z}_i^{*\prime}]$ are defined in Subsection 2.2. This, together with (A.14), results in

$$\frac{1}{n} \sum_{i=1}^{n} \psi_\tau(Y_i - \widehat{\theta}_2\mathbf{Z}_i^*)X_i = \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(Y_i - \theta_2\mathbf{Z}_i^*)(X_i - \Sigma_{21}'\Sigma_{22}^{-1}\mathbf{Z}_i^*) + o_p(n^{-1/2}). \quad \text{(A.15)}$$

Subsequently, by (A.12), (A.15), the central limit theorem, and the Cramer-Wold device, we obtain that

$$\sqrt{n} \begin{pmatrix} \widehat{\sigma}_{X|\mathbf{Z}}^2 - \sigma_{X|\mathbf{Z}}^2 \\ n^{-1}\sum_{i=1}^{n} \psi_\tau(Y_i - \widehat{\theta}_2\mathbf{Z}_i^*)X_i - E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X] \end{pmatrix} \rightarrow_d N(0, \Sigma_2),$$

where

$$\Sigma_2 = \begin{pmatrix} \Sigma_{23} & \Sigma_{25} \\ \Sigma_{25} & \Sigma_{24} \end{pmatrix},$$

and Σ_{23}, Σ_{24}, and Σ_{25} are defined as in Subsection 2.2. Finally, following the Delta method (van der Vaart, 1998, Chapter 3), we complete the proof.

Proof of Theorem 3. We first consider the term $\widetilde{\sigma}^2_{y|\mathbf{z}}$ in $\widetilde{\phi}_{kk,\tau}$. Let $\mathbf{z}^*_{t,k-1} = (1, \mathbf{z}'_{t,k-1})'$. Since $Ey_t^2 < \infty$ and $E[y_t - E(y_t|\mathcal{F}_{t-1})]^2 > 0$, the matrix $E(\mathbf{z}^*_{t,k-1}\mathbf{z}^{*\prime}_{t,k-1})$ is finite and positive definite. Analogous to (A.12), we can show that

$$\begin{aligned}\widetilde{\sigma}^2_{y|\mathbf{z}} &= \frac{1}{n}\sum_{t=k+1}^{n}(y_{t-k} - \alpha_1 - \beta_1'\mathbf{z}_{t,k-1})^2 + o_p(n^{-1/2}) \\ &= E(y_{t-k} - \alpha_1 - \beta_1'\mathbf{z}_{t,k-1})^2 + o_p(1). \qquad\qquad (A.16)\end{aligned}$$

We next study the numerator of $\widetilde{\phi}_{kk,\tau}$. Let $\theta_2 = (\phi_0(\tau), \phi_1(\tau), ..., \phi_p(\tau), \mathbf{0}')'$, and $\widetilde{\theta}_2 = (\widetilde{\alpha}_2, \widetilde{\beta}_2')'$, where $\mathbf{0}$ is the $(k-p)\times 1$ vector defined in the proof of Lemma 3, and $\widetilde{\alpha}_2$ and $\widetilde{\beta}_2$ are defined in Subsection 3.1. It is noteworthy that the series $\{y_t\}$ is fitted by model (3.1) with order $k-1$ and the true parameter vector θ_2. Accordingly, $e_{t,\tau} = y_t - \theta_2'\mathbf{z}^*_{t,k-1}$ and the parameter estimate of θ_2 is $\widetilde{\theta}_2$. Then, using (A.19) in the proof of Theorem 4, we obtain that

$$\sqrt{n}(\widetilde{\theta}_2 - \theta_2) = \{E[f_{t-1}(0)\mathbf{z}^*_{t,k-1}\mathbf{z}^{*\prime}_{t,k-1}]\}^{-1}\cdot\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(e_{t,\tau})\mathbf{z}^*_{t,k-1} + o_p(n^{-1/2}).$$

Applying a similar approach to that used in obtaining (A.9), and then using the above result, we further have that

$$\begin{aligned}&\frac{1}{n}\sum_{t=k+1}^{n}[\psi_\tau(y_t - \widetilde{\theta}_2'\mathbf{z}^*_{t-k}) - \psi_\tau(e_{t,\tau})]y_{t-k} \\ &\qquad = -\frac{1}{n}\sum_{t=k+1}^{n}\int_0^{(\widetilde{\theta}_2-\theta_2)'\mathbf{z}^*_{t,k-1}} f_{t-1}(s)ds\,y_{t-k} + o_p(n^{-1/2}) \\ &\qquad = -(\widetilde{\theta}_2 - \theta_2)'\cdot\frac{1}{n}\sum_{t=k+1}^{n}f_{t-1}(0)y_{t-k}\mathbf{z}^*_{t,k-1} + o_p(n^{-1/2}) \\ &\qquad = -A_1'\Sigma_{31}^{-1}\cdot\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(e_{t,\tau})\mathbf{z}^*_{t,k-1} + o_p(n^{-1/2}), \qquad\qquad (A.17)\end{aligned}$$

where A_1 and Σ_{31} are defined as in Subsection 3.1. Subsequently, using similar techniques to those for obtaining (A.5) and the result from equation (A.17), we obtain that

$$\begin{aligned}
&\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(y_t-\widetilde{\alpha}_2-\widetilde{\beta}_2'\mathbf{z}_{t,k-1})y_{t-k}\\
&\qquad=\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(e_{t,\tau})y_{t-k}+\frac{1}{n}\sum_{t=k+1}^{n}[\psi_\tau(y_t-\widetilde{\theta}_2'\mathbf{z}^*_{t,k-1})-\psi_\tau(e_{t,\tau})]y_{t-k}\\
&\qquad=\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(e_{t,\tau})[y_{t-k}-A_1'\Sigma_{31}^{-1}\mathbf{z}^*_{t,k-1}]+o_p(n^{-1/2}). \qquad \text{(A.18)}
\end{aligned}$$

Equations (A.16) and (A.18), together with the central limit theorem for the martingale difference sequence, complete the proof of the asymptotic normality of $\widetilde{\phi}_{kk,\tau}$. From Lemma 3, we also have that $\phi_{kk,\tau}=0$.

Proof of Theorem 4. For any $\mathbf{v}\in R^{p+1}$, denote

$$\begin{aligned}
Q(\mathbf{v})&=\sum_{t=p+1}^{n}\rho_\tau(y_t-(\boldsymbol{\phi}(\tau)+n^{-1/2}\mathbf{v})'\mathbf{z}^*_{t,p})-\sum_{t=p+1}^{n}\rho_\tau(y_t-\boldsymbol{\phi}'(\tau)\mathbf{z}^*_{t,p})\\
&=\sum_{t=p+1}^{n}\rho_\tau(e_{t,\tau}-n^{-1/2}\mathbf{v}'\mathbf{z}^*_{t,p})-\sum_{t=p+1}^{n}\rho_\tau(e_{t,\tau}),
\end{aligned}$$

where $e_{t,\tau}=y_t-\boldsymbol{\phi}'(\tau)\mathbf{z}^*_{t,p}$. Applying (A.1) and techniques similar to those in the proof of Theorem 3.1 in Koenker and Xiao (2006), we can show that

$$\begin{aligned}
Q(\mathbf{v})&=-\mathbf{v}'\cdot\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\psi_\tau(e_{t,\tau})\mathbf{z}^*_{t,p}+\sum_{t=p+1}^{n}\int_0^{n^{-1/2}\mathbf{v}'\mathbf{z}^*_{t,p}}I(e_{t,\tau}\le s)-I(e_{t,\tau}<0)ds\\
&=-\mathbf{v}'\cdot\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\psi_\tau(e_{t,\tau})\mathbf{z}^*_{t,p}+\frac{1}{2}\mathbf{v}'E[f_{t-1}(0)\mathbf{z}^*_{t,p}\mathbf{z}^{*\prime}_{t,p}]\mathbf{v}+o_p(1).
\end{aligned}$$

Note that $Q(\mathbf{v})$ is a convex function with respect to $\mathbf{v}$. By Knight (1998), we then have the Bahadur representation as follows,

$$\sqrt{n}\{\widetilde{\boldsymbol{\phi}}(\tau)-\boldsymbol{\phi}(\tau)\}=\{E[f_{t-1}(0)\mathbf{z}^*_{t,p}\mathbf{z}^{*\prime}_{t,p}]\}^{-1}\cdot\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\psi_\tau(e_{t,\tau})\mathbf{z}^*_{t,p}+o_p(1). \qquad \text{(A.19)}$$

This, in conjunction with the central limit theorem and the Cramer-Wold device, completes the proof.

Proof of Theorem 5. Without loss of generality, we assume that $\mathbf{z}_{1,p}$ is observable. Then

$$\widetilde{e}_{t,\tau}=y_t-\widetilde{\boldsymbol{\phi}}'(\tau)\mathbf{z}^*_{t,p}=y_t-\boldsymbol{\phi}'(\tau)\mathbf{z}^*_{t,p}-(\widetilde{\boldsymbol{\phi}}(\tau)-\boldsymbol{\phi}(\tau))'\mathbf{z}^*_{t,p}=e_{t,\tau}-(\widetilde{\boldsymbol{\phi}}(\tau)-\boldsymbol{\phi}(\tau))'\mathbf{z}^*_{t,p}$$

for $1 \leq t \leq n$. We first consider the term $\widetilde{\sigma}_e^2$ in $r_{k,\tau}$. By the ergodic theorem and the fact that $\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau) = O_p(n^{-1/2})$, we can show that

$$\widetilde{\mu}_e = \frac{1}{n}\sum_{t=k+1}^{n} \widetilde{e}_{t,\tau} = \frac{1}{n}\sum_{t=k+1}^{n} e_{t,\tau} - (\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau))'\frac{1}{n}\sum_{t=k+1}^{n} \mathbf{z}_{t,p}^* = E(e_{t,\tau}) + o_p(1),$$

and

$$\begin{aligned}\widetilde{\sigma}_e^2 &= \frac{1}{n}\sum_{t=k+1}^{n} (\widetilde{e}_{t,\tau} - \widetilde{\mu}_e)^2 = \frac{1}{n}\sum_{t=k+1}^{n} \widetilde{e}_{t,\tau}^2 - \widetilde{\mu}_e^2 \\ &= \frac{1}{n}\sum_{t=k+1}^{n} e_{t,\tau}^2 - 2(\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau))' \cdot \frac{1}{n}\sum_{t=k+1}^{n} e_{t,\tau}\mathbf{z}_{t,p}^* \\ &\qquad + (\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau))' \cdot \frac{1}{n}\sum_{t=k+1}^{n} \mathbf{z}_{t,p}^*\mathbf{z}_{t,p}^{*\prime} \cdot (\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau)) - \widetilde{\mu}_e^2 \\ &= \sigma_e^2 + o_p(1), \end{aligned} \tag{A.20}$$

where σ_e^2 is defined in Subsection 3.2.

We next consider the numerator of $r_{k,\tau}$. Using the fact that $|\sum_{t=k+1}^{n} \psi_\tau(\widetilde{e}_{t,\tau})| < 1$, we obtain

$$\begin{aligned}&\frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(\widetilde{e}_{t,\tau})(\widetilde{e}_{t-k,\tau} - \widetilde{\mu}_e) \\ &\quad = \frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\boldsymbol{\phi}}'(\tau)\mathbf{z}_{t,p}^*)[e_{t-k,\tau} - (\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau))'\mathbf{z}_{t-k,p}^*] + O_p(n^{-1}) \\ &\quad = \frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\boldsymbol{\phi}}'(\tau)\mathbf{z}_{t,p}^*)e_{t-k,\tau} \\ &\qquad - (\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau))' \cdot \frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\boldsymbol{\phi}}'(\tau)\mathbf{z}_{t,p}^*)\mathbf{z}_{t-k,p}^* + o_p(n^{-1/2}). \end{aligned} \tag{A.21}$$

Applying similar techniques to those used in obtaining (A.9), we are able to show that

$$\begin{aligned}&\frac{1}{n}\sum_{t=k+1}^{n} [\psi_\tau(y_t - \widetilde{\boldsymbol{\phi}}'(\tau)\mathbf{z}_{t,p}^*) - \psi_\tau(e_{t,\tau})]e_{t-k,\tau} \\ &\quad = -\Sigma_{51,k}\Sigma_{41}^{-1} \cdot \frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(e_{t,\tau})\mathbf{z}_{t,p}^* + o_p(n^{-1/2}),\end{aligned}$$

where Σ_{41} is defined in Subsection 3.1 and $\Sigma_{51,k} = E[f_{t-1}(0)e_{t-k,\tau}\mathbf{z}_{t,p}^{*\prime}]$. In addition, using similar techniques to those in obtaining (A.5) and the above result, we further obtain that

$$\frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\boldsymbol{\phi}}'(\tau)\mathbf{z}_{t,p}^*)e_{t-k,\tau} = \frac{1}{n}\sum_{t=k+1}^{n} \psi_\tau(e_{t,\tau})[e_{t-k,\tau} - \Sigma_{51,k}\Sigma_{41}^{-1}\mathbf{z}_{t,p}^*] + o_p(n^{-1/2}).$$

Analogously, we can verify that

$$\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(y_t-\widetilde{\boldsymbol{\phi}}'(\tau)\mathbf{z}_{t,p}^*)\mathbf{z}_{t-k,p}^* = O_p(n^{-1/2}).$$

The above results, together with (A.20), (A.21), and the fact that $\widetilde{\boldsymbol{\phi}}(\tau)-\boldsymbol{\phi}(\tau)=O_p(n^{-1/2})$, imply

$$r_{k,\tau}=\frac{1}{\sqrt{(\tau-\tau^2)\sigma_e^2}}\cdot\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(e_{t,\tau})[e_{t-k,\tau}-\Sigma_{51,k}\Sigma_{41}^{-1}\mathbf{z}_{t,p}^*]+o_p(n^{-1/2}),$$

and

$$R_\tau=\frac{1}{\sqrt{(\tau-\tau^2)\sigma_e^2}}\cdot\frac{1}{n}\sum_{t=k+1}^{n}\psi_\tau(e_{t,\tau})[\mathbf{e}_{t-1,K}-\Sigma_{51}\Sigma_{41}^{-1}\mathbf{z}_{t,p}^*]+o_p(n^{-1/2}),$$

where $\mathbf{e}_{t-1,K}$ and Σ_{51} are defined in Subsection 3.3. Subsequently, applying the central limit theorem for the martingale difference sequence and the Cramer-Wold device, we complete the proof.

Proof of Theorem 6. Let $\mathbf{z}_{t,p}^*=(1,\mathbf{z}_{t,p}')'$. Since

$$\Sigma_{41}^*=\frac{1}{n}\sum_{t=p+1}^{n}\omega_t f_{t-1}(0)\mathbf{z}_{t,p}^*\mathbf{z}_{t,p}^{*\prime}=E[f_{t-1}(0)\mathbf{z}_{t,p}^*\mathbf{z}_{t,p}^{*\prime}]+o_p^*(1),$$

we can apply a method similar to the proof of Lemma 2.2 in Rao and Zhao (1992) and that of Theorem 4 to show that

$$\sqrt{n}\{\widetilde{\boldsymbol{\phi}}^*(\tau)-\boldsymbol{\phi}(\tau)\}=(\Sigma_{41}^*)^{-1}\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}\omega_t\psi_\tau(e_{t,\tau})\mathbf{z}_{t,p}^*+o_p^*(1), \tag{A.22}$$

where notations $o_p^*(1)$ is referred to the bootstrapped probability space. This, together with (A.19), implies

$$\sqrt{n}\{\widetilde{\boldsymbol{\phi}}^*(\tau)-\widetilde{\boldsymbol{\phi}}(\tau)\}=\{E[f_{t-1}(0)\mathbf{z}_{t,p}^*\mathbf{z}_{t,p}^{*\prime}]\}^{-1}\frac{1}{\sqrt{n}}\sum_{t=p+1}^{n}(\omega_t-1)\psi_\tau(e_{t,\tau})\mathbf{z}_{t,p}^*+o_p^*(1). \tag{A.23}$$

By the central limit theorem for the martingale difference sequence, we complete the proof of part (b).

Using the results of (A.17) and (A.18), for any $k>p$, we can demonstrate that

$$\frac{1}{\sqrt{n}}\sum_{t=k+1}^{n}\psi_\tau(y_t-\widetilde{\alpha}_2-\widetilde{\beta}_2'\mathbf{z}_{t,k-1})y_{t-k}$$
$$=\frac{1}{\sqrt{n}}\sum_{t=k+1}^{n}\psi_\tau(e_{t,\tau})y_{t-k}-\sqrt{n}(\widetilde{\theta}_2-\theta_2)'A_1+o_p(1),$$

where $\theta_2 = (\alpha_2, \beta_2')'$ and $\widetilde{\theta}_2 = (\widetilde{\alpha}_2, \widetilde{\beta}_2')'$. In addition, applying a method similar to the proof of (A.22), we are able to show that

$$\frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \omega_t \psi_\tau(y_t - \widetilde{\alpha}_2^* - \widetilde{\beta}_2^{*\prime} \mathbf{z}_{t,k-1}) y_{t-k} = \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \omega_t \psi_\tau(e_{t,\tau}) y_{t-k} - \sqrt{n}(\widetilde{\theta}_2^* - \theta_2)' A_1 + o_p^*(1),$$

where $\widetilde{\theta}_2^* = (\widetilde{\alpha}_2^*, \widetilde{\beta}_2^{*\prime})'$. As a result,

$$\frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \omega_t \psi_\tau(y_t - \widetilde{\alpha}_2^* - \widetilde{\beta}_2^{*\prime} \mathbf{z}_{t,k-1}) y_{t-k} - \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\alpha}_2 - \widetilde{\beta}_2' \mathbf{z}_{t,k-1}) y_{t-k} = \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} (\omega_t - 1) \psi_\tau(e_{t,\tau}) y_{t-k} - \sqrt{n}(\widetilde{\theta}_2^* - \widetilde{\theta}_2)' A_1 + o_p^*(1). \quad \text{(A.24)}$$

In addition, for $k > p$, (A.23) implies that

$$\sqrt{n}(\widetilde{\theta}_2^* - \widetilde{\theta}_2) = \Sigma_{31}^{-1} \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} (\omega_t - 1) \psi_\tau(e_{t,\tau}) \mathbf{z}_{t,k-1}^* + o_p^*(1). \quad \text{(A.25)}$$

By (A.24) and (A.25), we have that

$$\begin{aligned} \sqrt{(\tau - \tau^2)\widetilde{\sigma}_{y|\mathbf{z}}^2} \cdot \sqrt{n}(\widetilde{\phi}_{kk,\tau}^* - \widetilde{\phi}_{kk,\tau}) &= \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \omega_t \psi_\tau(y_t - \widetilde{\alpha}_2^* - \widetilde{\beta}_2^{*\prime} \mathbf{z}_{t,k-1}) y_{t-k} \\ &\quad - \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\alpha}_2 - \widetilde{\beta}_2' \mathbf{z}_{t,k-1}) y_{t-k} \\ &= \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} (\omega_t - 1) \psi_\tau(e_{t,\tau}) [y_{t-k} - A_1' \Sigma_{31}^{-1} \mathbf{z}_{t,k-1}^*] + o_p^*(1). \end{aligned}$$

This, in conjunction with (A.16) and the central limit theorem for the martingale difference sequence, completes the proof of part (a).

To prove part (c), we assume that, without loss of generality, $k \geq p$. From the proof of Theorem 5, we have that

$$\frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \psi_\tau(\widetilde{e}_{t,\tau}) \widetilde{e}_{t-k,\tau} = \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \psi_\tau(e_{t,\tau}) e_{t-k,\tau} - \Sigma_{51,k} \cdot \sqrt{n}(\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau)) + o_p(1). \quad \text{(A.26)}$$

Let $\widetilde{e}_{t,\tau}^* = y_t - \widetilde{\boldsymbol{\phi}}^{*\prime}(\tau) \mathbf{z}_{t,p}$. By a method similar to the proof of (A.22), we can show that

$$\frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \omega_t \psi_\tau(\widetilde{e}_{t,\tau}^*) \widetilde{e}_{t-k,\tau}^* = \frac{1}{\sqrt{n}} \sum_{t=k+1}^{n} \omega_t \psi_\tau(e_{t,\tau}) e_{t-k,\tau} - \Sigma_{51,k} \cdot \sqrt{n}(\widetilde{\boldsymbol{\phi}}^*(\tau) - \boldsymbol{\phi}(\tau)) + o_p^*(1). \quad \text{(A.27)}$$

Note that $|\sum_{t=k+1}^{n}\psi_\tau(\widetilde{e}_{t,\tau})| < 1$ and $\widetilde{\mu}_e = n^{-1}\sum_{t=k+1}^{n}\widetilde{e}_{t,\tau} = O_p(1)$. This, together with (A.23), (A.26) and (A.27), implies that

$$\begin{aligned}
&\sqrt{(\tau-\tau^2)\widetilde{\sigma}_e^2}\cdot\sqrt{n}(r_{k,\tau}^* - r_{k,\tau}) \\
&\qquad = \frac{1}{\sqrt{n}}\sum_{t=k+1}^{n}\omega_t\psi_\tau(\widetilde{e}_{t,\tau}^*)\widetilde{e}_{t-k,\tau}^* - \frac{1}{\sqrt{n}}\sum_{t=k+1}^{n}\psi_\tau(\widetilde{e}_{t,\tau})(\widetilde{e}_{t-k,\tau} - \widetilde{\mu}_e) \\
&\qquad = \frac{1}{\sqrt{n}}\sum_{t=k+1}^{n}(\omega_t - 1)\psi_\tau(e_{t,\tau})[e_{t-k,\tau} - \Sigma_{51,k}\Sigma_{41}^{-1}\mathbf{z}_{t,p}^*] + o_p^*(1).
\end{aligned}$$

Then, applying similar techniques to those used in the proofs of parts (a), (b) and Theorem 5, in conjunction with (A.20), we complete the proof of part (c).

References

Ando, T. and Tsay, R. S. (2011). Quantile regression models with factor-augmented predictors and information criterion. *Econometrics Journal*, 14:1–24.

Baur, D. G., Dimpfl, T., and Jung, R. C. (2012). Stock return autocorrelations revisited: A quantile regression approach. *Journal of Empirical Finance*, 19:251–265.

Bofinger, E. (1975). Estimation of a density function using order statistics. *Australian Journal of Statistics*, 17:1–7.

Box, G. E. P., Jenkins, G. M., and Reinsel, G. C. (2008). *Time Series Analysis, Forecasting and Control*. Wiley, New York, 4th edition.

Box, G. E. P. and Pierce, D. A. (1970). Distribution of the residual autocorrelations in autoregressive integrated moving average time series models. *Journal of the American Statistical Association*, 65:1509–1526.

Cai, Y., Stander, J., and Davies, N. (2012). A new bayesian approach to quantile autoregressive time series model estimation and forecasting. *Journal of Time Series Analysis*, 33:684–698.

Chatterjee, S. and Hadi, A. S. (2006). *Regression Analysis by Example*. John Wiley & Sons, New York, 4th edition.

Fan, J. and Lv, J. (2008). Sure independence screening for ultrahigh dimensional feature space. *Journal of the Royal Statistical Society, Series B*, 70:849–911.

Fan, J. and Yao, Q. (2003). *Nonlinear Time Series: Nonparametric and Parametric Methods*. Springer, New York.

Feng, X. and He, X. (2011). Wild bootstrap for quantile regression. *Biometrika*, 98:995–999.

Galvao, A. F., Montes-Rojas, G., and Park, S. Y. (2012). Quantile autoregressive distributed lag model with an application to house price returns. *Oxford Bulletin of Economics and Statistics*, pages DOI: 10.1111/j.1468–0084.2011.00683.x.

Hall, P. and Sheather, S. (1988). On the distribution of a studentized quantile. *Journal of the Royal Statistical Society, Series B*, 50:381–391.

He, X. and Hu, F. (2002). Markov chain marginal bootstrap. *Journal of the American Statistical Association*, 97:783–795.

He, X. and Liang, H. (2000). Quantile regression estimates for a class of linear and partially linear errors-in-variables models. *Statistica Sinica*, 10:129–140.

He, X. and Zhu, L.-X. (2003). A lack-of-fit test for quantile regression. *Journal of the American Statistical Association*, 98:1013–1022.

Hendricks, W. and Koenker, R. (1991). Hierarchical spline models for conditional quantiles and the demand for electricity. *Journal of the American Statistical Association*, 87:58–68.

Jin, Z., Ying, Z., and Wei, L. J. (2001). A simple resampling method by perturbing the minimand. *Biometrika*, 88:381–390.

Kai, B., Li, R., and Zou, H. (2011). New efficient estimation and variable selection methods for semiparametric varying-coefficient partially linear models. *The Annals of Statistics*, 39:305–332.

Knight, K. (1998). Limiting distributions for l_1 regression estimators under general conditions. *The Annals of Statistics*, 26:755–770.

Kocherginsky, M., He, X., and Mu, Y. (2005). Practical confidence intervals for regression quantiles. *Journal of Computational and Graphical Statistics*, 14:41–55.

Koenker, R. (2005). *Quantile regression*. Cambridge University Press, Cambridge.

Koenker, R. and Bassett, G. (1978). Regression quantiles. *Econometrica*, 46:33–49.

Koenker, R. and Xiao, Z. (2006). Quantile autoregression. *Journal of the American Statistical Association*, 101:980–990.

Koul, H. L. and Saleh, A. (1995). Autoregression quantiles and related rank scores processes. *The Annals of Statistics*, 23:670–689.

Li, G., Leng, C., and Tsai, C.-L. (2012). A hybrid bootstrap approach to unit root tests. *Journal of Time Series Analysis*, In press.

Li, W. K. (2004). *Diagnostic Checks in Time Series*. Chapman & Hall, Boca Raton.

Ling, S. and McAleer, M. (2004). Regression quantiles for unstable autoregressive models. *Journal of Multivariate Analysis*, 89:304–328.

Machado, J. A. F. and Silva, J. M. C. S. (2005). Quantile for counts. *Journal of the American Statistical Association*, 100:1226–1237.

Mu, Y. and He, X. (2007). Power transfromation toward a linear regression quantile. *Journal of the American Statistical Association*, 102:269–279.

Peng, L. and Huang, Y. (2008). Survival analysis with quantile regression models. *Journal of the American Statistical Association*, 103:637–649.

Rao, C. R. and Zhao, L. C. (1992). Approximation to the distribution of m-estimates in linear models by randomly weighted bootstrap. *Sankhya*, 54:323–331.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. Wiley, New York.

van der Vaart, A. W. (1998). *Asymptotic Statistics*. Cambridge University Press, Cambridge.

Veronesi, P. (1999). Stock market overreactions to bad news in good times: a rational expectations equilibrium model. *The Review of Financial Studies*, 12:975–1007.

Wang, H. (2009). Forward regression for ultra-high dimensional variable screening. *Journal of the American Statistical Association*, 104:1512–1524.

Wang, H. J. and Wang, L. (2009). Locally weighted censored quantile regression. *Journal of the American Statistical Association*, 104:1117–1128.

Wei, Y. and Carroll, R. J. (2009). Quantile regression with measurement error. *Journal of the American Statistical Association*, 104:1129–1143.

Xiao, Z. and Koenker, R. (2009). Conditional quantile estimation for generalized autoregressive conditional heteroscedasticity models. *Journal of the American Statistical Association*, 104:1696–1712.

Table 1: Bias (BIAS), estimated standard deviation (ESD), and asymptotic standard deviation (ASD) of the sample quantile correlation $\widehat{\mathrm{qcor}}_\tau\{Y, X\}$.

n	τ	BIAS	ESD	ASD			
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$
100	0.25	-0.0045	0.0828	0.0867	0.0860	0.0836	0.0891
	0.50	-0.0032	0.0792	0.0835	0.0829	0.0816	0.0847
	0.75	-0.0029	0.0824	0.0873	0.0863	0.0836	0.0897
200	0.25	-0.0012	0.0598	0.0601	0.0596	0.0586	0.0607
	0.50	-0.0000	0.0585	0.0580	0.0577	0.0571	0.0582
	0.75	0.0005	0.0562	0.0601	0.0596	0.0586	0.0608
500	0.25	0.0006	0.0365	0.0372	0.0369	0.0367	0.0373
	0.50	0.0002	0.0361	0.0362	0.0361	0.0359	0.0362
	0.75	-0.0005	0.0367	0.0371	0.0369	0.0367	0.0373

Table 2: Bias (BIAS), estimated standard deviation (ESD), and asymptotic standard deviation (ASD) of the sample quantile partial correlation $\widehat{\mathrm{qpcor}}_\tau\{Y, X|Z\}$.

n	τ	BIAS	ESD	ASD			
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$
100	0.25	-0.0094	0.0901	0.0972	0.0963	0.0922	0.1004
	0.50	-0.0026	0.0931	0.0943	0.0935	0.0912	0.0959
	0.75	0.0046	0.0971	0.0974	0.0961	0.0921	0.1008
200	0.25	-0.0052	0.0663	0.0677	0.0669	0.0651	0.0688
	0.50	-0.0004	0.0654	0.0660	0.0654	0.0646	0.0664
	0.75	0.0022	0.0655	0.0677	0.0669	0.0654	0.0689
500	0.25	-0.0006	0.0420	0.0421	0.0417	0.0412	0.0423
	0.50	0.0001	0.0407	0.0413	0.0411	0.0408	0.0413
	0.75	0.0004	0.0403	0.0421	0.0417	0.0413	0.0423

Table 3: Bias (BIAS), estimated standard deviation (ESD), and asymptotic standard deviation (ASD) of $\widetilde{\phi}_{kk,\tau}$, computed from the time series data with $i.i.d.$ conditional quantile errors, at lags $k = 2$, 3, and 4.

n	k	BIAS	ESD	ASD			
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$
				$\tau = 0.25$			
100	2	-0.0173	0.0993	0.1001	0.1001	0.1001	0.1002
	3	-0.0047	0.1042	0.1003	0.1002	0.1001	0.1007
	4	-0.0135	0.0980	0.1007	0.1006	0.1002	0.1014
200	2	-0.0109	0.0710	0.0707	0.0707	0.0707	0.0707
	3	-0.0035	0.0698	0.0707	0.0707	0.0707	0.0708
	4	-0.0039	0.0698	0.0709	0.0708	0.0707	0.0710
500	2	-0.0055	0.0460	0.0447	0.0447	0.0447	0.0447
	3	-0.0010	0.0454	0.0447	0.0447	0.0447	0.0447
	4	-0.0025	0.0458	0.0447	0.0447	0.0447	0.0447
				$\tau = 0.5$			
100	2	-0.0188	0.1030	0.1001	0.1001	0.1001	0.1001
	3	-0.0086	0.0997	0.1001	0.1001	0.1001	0.1002
	4	-0.0178	0.0995	0.1004	0.1003	0.1003	0.1006
200	2	-0.0119	0.0709	0.0707	0.0707	0.0707	0.0707
	3	-0.0014	0.0690	0.0707	0.0707	0.0707	0.0707
	4	-0.0078	0.0705	0.0708	0.0708	0.0707	0.0708
500	2	-0.0044	0.0456	0.0447	0.0447	0.0447	0.0447
	3	0.0007	0.0435	0.0447	0.0447	0.0447	0.0447
	4	-0.0047	0.0456	0.0447	0.0447	0.0447	0.0447
				$\tau = 0.75$			
100	2	-0.0156	0.1032	0.1001	0.1001	0.1001	0.1003
	3	-0.0073	0.1049	0.1003	0.1002	0.1001	0.1005
	4	-0.0160	0.1048	0.1007	0.1007	0.1003	0.1014
200	2	-0.0083	0.0699	0.0707	0.0707	0.0707	0.0707
	3	-0.0031	0.0693	0.0708	0.0707	0.0707	0.0708
	4	-0.0109	0.0745	0.0709	0.0708	0.0708	0.0710
500	2	-0.0025	0.0442	0.0447	0.0447	0.0447	0.0447
	3	-0.0002	0.0451	0.0447	0.0447	0.0447	0.0447
	4	-0.0058	0.0454	0.0447	0.0447	0.0447	0.0447

Table 4: Bias (BIAS), estimated standard deviation (ESD), and asymptotic standard deviation (ASD) of $\widetilde{\phi}_k(\tau)$, computed from the time series data with $i.i.d.$ conditional quantile errors, at lags $k = 0$ and 1.

n	k	BIAS	ESD	ASD			
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$
				$\tau = 0.25$			
100	0	0.0016	0.1396	0.1535	0.1573	0.1291	0.1492
	1	-0.0187	0.1208	0.1284	0.1320	0.1109	0.1220
200	0	-0.0010	0.0969	0.1029	0.1059	0.0914	0.1012
	1	-0.0068	0.0838	0.0868	0.0901	0.0787	0.0842
500	0	0.0026	0.0625	0.0633	0.0648	0.0597	0.0629
	1	-0.0038	0.0549	0.0539	0.0555	0.0514	0.0534
				$\tau = 0.5$			
100	0	-0.0039	0.1290	0.1342	0.1380	0.1373	0.1320
	1	-0.0186	0.1151	0.1144	0.1184	0.1177	0.1109
200	0	-0.0046	0.0916	0.0921	0.0947	0.1859	0.0914
	1	-0.0110	0.0764	0.0786	0.0816	0.1587	0.0776
500	0	-0.0001	0.0575	0.0574	0.0584	0.0679	0.0572
	1	-0.0043	0.0482	0.0492	0.0504	0.0589	0.0490
				$\tau = 0.75$			
100	0	-0.0118	0.1445	0.1511	0.1553	0.1278	0.1466
	1	-0.0197	0.1227	0.1261	0.1307	0.1098	0.1195
200	0	-0.0099	0.0999	0.1030	0.1062	0.0916	0.1013
	1	-0.0104	0.0839	0.0868	0.0903	0.0790	0.0844
500	0	-0.0031	0.0624	0.0627	0.0644	0.0594	0.0624
	1	-0.0047	0.0515	0.0533	0.0552	0.0513	0.0529

Table 5: Bias (BIAS), estimated standard deviation (ESD) and asymptotic standard deviation (ASD) of $r_{k,\tau}$, computed from the time series data with $i.i.d.$ conditional quantile errors, at lags $k = 2$, 4, and 6.

n	QACF	BIAS	ESD	ASD			
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$
				$\tau = 0.25$			
100	$r_{2,\tau}$	-0.0039	0.0947	0.0902	0.0902	0.0902	0.0903
	$r_{4,\tau}$	-0.0021	0.1023	0.0987	0.0987	0.0986	0.0988
	$r_{6,\tau}$	0.0055	0.1090	0.0991	0.0991	0.0990	0.0992
200	$r_{2,\tau}$	-0.0052	0.0651	0.0637	0.0637	0.0637	0.0637
	$r_{4,\tau}$	0.0005	0.0728	0.0700	0.0700	0.0700	0.0700
	$r_{6,\tau}$	-0.0016	0.0738	0.0703	0.0703	0.0703	0.0703
500	$r_{2,\tau}$	-0.0024	0.0407	0.0403	0.0403	0.0403	0.0403
	$r_{4,\tau}$	-0.0006	0.0452	0.0443	0.0443	0.0443	0.0443
	$r_{6,\tau}$	-0.0003	0.0445	0.0446	0.0446	0.0446	0.0446
				$\tau = 0.5$			
100	$r_{2,\tau}$	-0.0063	0.0932	0.0901	0.0901	0.0901	0.0902
	$r_{4,\tau}$	-0.0154	0.0979	0.0986	0.0986	0.0986	0.0986
	$r_{6,\tau}$	-0.0073	0.1036	0.0990	0.0990	0.0990	0.0991
200	$r_{2,\tau}$	-0.0031	0.0629	0.0636	0.0636	0.0636	0.0637
	$r_{4,\tau}$	-0.0055	0.0700	0.0700	0.0700	0.0700	0.0700
	$r_{6,\tau}$	-0.0060	0.0716	0.0703	0.0703	0.0703	0.0703
500	$r_{2,\tau}$	-0.0010	0.0405	0.0403	0.0403	0.0403	0.0403
	$r_{4,\tau}$	-0.0031	0.0444	0.0443	0.0443	0.0443	0.0443
	$r_{6,\tau}$	-0.0015	0.0435	0.0446	0.0446	0.0446	0.0446
				$\tau = 0.75$			
100	$r_{2,\tau}$	-0.0037	0.0949	0.0902	0.0902	0.0902	0.0903
	$r_{4,\tau}$	-0.0188	0.1022	0.0987	0.0987	0.0986	0.0988
	$r_{6,\tau}$	-0.0163	0.1080	0.0991	0.0991	0.0990	0.0992
200	$r_{2,\tau}$	-0.0032	0.0655	0.0637	0.0637	0.0637	0.0637
	$r_{4,\tau}$	-0.0087	0.0683	0.0700	0.0700	0.0700	0.0700
	$r_{6,\tau}$	-0.0104	0.0729	0.0703	0.0703	0.0703	0.0703
500	$r_{2,\tau}$	-0.0006	0.0397	0.0403	0.0403	0.0403	0.0403
	$r_{4,\tau}$	-0.0041	0.0447	0.0443	0.0443	0.0443	0.0443
	$r_{6,\tau}$	-0.0042	0.0460	0.0446	0.0446	0.0446	0.0446

Table 6: Rejection rate of the test statistic $Q_{BP}(K)$ with $K = 6$, $\tau = 0.25$, 0.5 and 0.75, and the 5% nominal significance level.

		i.i.d. errors			Non-*i.i.d.* errors		
n	ϕ	0.25	0.5	0.75	0.25	0.5	0.75
100	0.0	0.060	0.051	0.055	0.070	0.064	0.077
	0.1	0.077	0.095	0.088	0.446	0.165	0.130
	0.2	0.177	0.183	0.173	0.696	0.508	0.208
200	0.0	0.054	0.048	0.053	0.058	0.055	0.062
	0.1	0.102	0.101	0.106	0.776	0.256	0.153
	0.2	0.309	0.330	0.285	0.957	0.796	0.314
500	0.0	0.054	0.051	0.046	0.051	0.050	0.048
	0.1	0.199	0.227	0.175	0.990	0.557	0.172
	0.2	0.705	0.780	0.683	0.999	0.982	0.521

Table 7: Bias (BIAS), estimated standard deviation (ESD), and asymptotic standard deviation (ASD) of $\widetilde{\phi}_{kk,\tau}$, computed from the time series data with non-$i.i.d.$ conditional quantile errors, at lags $k = 2$, 3 and 4 for $\tau = 0.25$, and $k = 3$ and 4 for $\tau = 0.5$ and 0.75.

n	k	BIAS	ESD	ASD				
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$	Boot
				$\tau = 0.25$				
100	2	-0.0220	0.1010	0.1021	0.1019	0.1013	0.1025	0.1082
	3	-0.0205	0.1058	0.1059	0.1058	0.1038	0.1064	0.1107
	4	-0.0208	0.1064	0.1059	0.1054	0.1035	0.1068	0.1129
200	2	-0.0134	0.0721	0.0716	0.0714	0.0711	0.0717	0.0750
	3	-0.0089	0.0732	0.0732	0.0730	0.0723	0.0733	0.0765
	4	-0.0119	0.0741	0.0729	0.0724	0.0717	0.0731	0.0776
500	2	-0.0055	0.0446	0.0451	0.0451	0.0449	0.0451	0.0467
	3	-0.0051	0.0451	0.0460	0.0459	0.0457	0.0459	0.0476
	4	-0.0033	0.0473	0.0457	0.0454	0.0451	0.0459	0.0476
				$\tau = 0.5$				
100	3	-0.0110	0.1062	0.1038	0.1035	0.1036	0.1045	0.1153
	4	-0.0215	0.1043	0.1035	0.1034	0.1034	0.1044	0.1176
200	3	-0.0019	0.0722	0.0722	0.0722	0.0722	0.0723	0.0792
	4	-0.0101	0.0729	0.0718	0.0717	0.0720	0.0719	0.0799
500	3	-0.0035	0.0461	0.0456	0.0456	0.0455	0.0456	0.0484
	4	-0.0009	0.0463	0.0451	0.0450	0.0450	0.0451	0.0484
				$\tau = 0.75$				
100	3	-0.0117	0.1062	0.1037	0.1038	0.1035	0.1040	0.1183
	4	-0.0192	0.1052	0.1045	0.1043	0.1031	0.1051	0.1218
200	3	-0.0054	0.0728	0.0721	0.0720	0.0720	0.0723	0.0800
	4	-0.0096	0.0744	0.0721	0.0719	0.0716	0.0723	0.0813
500	3	-0.0042	0.0455	0.0456	0.0455	0.0455	0.0456	0.0484
	4	-0.0013	0.0448	0.0452	0.0451	0.0450	0.0452	0.0489

Table 8: Bias (BIAS), estimated standard deviation (ESD), and asymptotic standard deviation (ASD) of $\widetilde{\phi}_k(\tau)$, computed from the time series data with non-$i.i.d.$ conditional quantile errors, at lags $k = 0$, 1 for $\tau = 0.25$ and $k = 0$, 1 and 2 for $\tau = 0.5$ and 0.75.

n	k	BIAS	ESD	ASD				
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$	Boot
				$\tau = 0.25$				
100	0	0.0012	0.0598	0.0827	0.0835	0.1708	0.0790	0.0818
	1	0.0067	0.0266	0.0320	0.0319	0.0614	0.0315	0.0333
200	0	-0.0022	0.0370	0.0505	0.0535	0.0943	0.0459	0.0438
	1	0.0040	0.0166	0.0196	0.0202	0.0350	0.0185	0.0194
500	0	-0.0009	0.0221	0.0255	0.0307	0.0477	0.0239	0.0238
	1	0.0015	0.0080	0.0098	0.0113	0.0179	0.0092	0.0098
				$\tau = 0.5$				
100	0	0.0610	0.2571	0.2865	0.3026	0.3005	0.2747	0.2702
	1	0.0082	0.0788	0.0904	0.0954	0.0943	0.0883	0.0913
	2	-0.0270	0.1217	0.1373	0.1445	0.1437	0.1336	0.1365
200	0	0.0316	0.1703	0.1899	0.2006	0.6124	0.1842	0.1831
	1	0.0038	0.0509	0.0585	0.0607	0.1817	0.0573	0.0599
	2	-0.0142	0.0931	0.1020	0.1056	0.2992	0.0981	0.0992
500	0	0.0109	0.1106	0.1149	0.1240	0.1592	0.1133	0.1143
	1	0.0022	0.0331	0.0336	0.0359	0.0462	0.0331	0.0347
	2	-0.0068	0.0618	0.0635	0.0685	0.0859	0.0625	0.0632
				$\tau = 0.75$				
100	0	0.2260	0.5369	0.6815	0.7204	0.4109	0.6190	0.5977
	1	0.0006	0.1629	0.2165	0.2235	0.1250	0.2094	0.1889
	2	-0.1030	0.2486	0.3297	0.3479	0.1957	0.2994	0.2886
200	0	0.1105	0.4031	0.4349	0.4644	0.3147	0.4144	0.4090
	1	0.0033	0.1109	0.1373	0.1463	0.0950	0.1345	0.1306
	2	-0.0550	0.1960	0.2307	0.2464	0.1655	0.2204	0.2191
500	0	0.0527	0.2428	0.2636	0.2794	0.2272	0.2588	0.2607
	1	0.0018	0.0725	0.0792	0.0822	0.0662	0.0785	0.0790
	2	-0.0272	0.1337	0.1439	0.1512	0.1223	0.1419	0.1424

Table 9: Bias (BIAS), estimated standard deviation (ESD) and asymptotic standard deviation (ASD) of $r_{k,\tau}$, computed from the time series data with non-$i.i.d.$ conditional quantile errors, at lags $k = 2$, 4, and 6.

n	QACF	BIAS	ESD	ASD				
				h_{HS}	h_B	$3h_{HS}$	$0.6h_B$	Boot
				$\tau = 0.25$				
100	$r_{2,\tau}$	-0.0617	0.9875	0.9663	0.9641	0.9571	0.9700	1.1083
	$r_{4,\tau}$	-0.0101	1.0146	0.9839	0.9846	0.9813	0.9860	1.1380
	$r_{6,\tau}$	0.0418	1.0597	0.9917	0.9911	0.9894	0.9930	1.1420
200	$r_{2,\tau}$	-0.0518	0.9688	0.9603	0.9622	0.9532	0.9633	1.0637
	$r_{4,\tau}$	-0.0046	1.0288	0.9818	0.9821	0.9798	0.9835	1.0907
	$r_{6,\tau}$	0.0066	1.0255	0.9923	0.9921	0.9908	0.9925	1.0904
500	$r_{2,\tau}$	-0.0612	0.9707	0.9562	0.9557	0.9519	0.9561	1.0275
	$r_{4,\tau}$	-0.0111	1.0262	0.9828	0.9823	0.9806	0.9832	1.0459
	$r_{6,\tau}$	-0.0040	1.0085	0.9944	0.9939	0.9932	0.9944	1.0523
				$\tau = 0.5$				
100	$r_{2,\tau}$	0.1671	0.5818	0.3898	0.3897	0.3897	0.3901	0.9138
	$r_{4,\tau}$	-0.0258	1.0051	0.9490	0.9488	0.9488	0.9500	1.1744
	$r_{6,\tau}$	-0.0042	1.0476	0.9833	0.9831	0.9832	0.9844	1.1893
200	$r_{2,\tau}$	0.1448	0.5059	0.3772	0.3766	0.3776	0.3775	0.7485
	$r_{4,\tau}$	-0.0578	0.9750	0.9431	0.9427	0.9441	0.9433	1.0935
	$r_{6,\tau}$	-0.0255	1.0367	0.9851	0.9846	0.9860	0.9858	1.1113
500	$r_{2,\tau}$	0.0784	0.4494	0.3716	0.3715	0.3709	0.3717	0.5827
	$r_{4,\tau}$	0.0112	0.9424	0.9404	0.9403	0.9402	0.9406	1.0221
	$r_{6,\tau}$	-0.0125	0.9966	0.9866	0.9868	0.9865	0.9867	1.0590
				$\tau = 0.75$				
100	$r_{2,\tau}$	0.1255	0.6919	0.5459	0.5482	0.5433	0.5475	1.0945
	$r_{4,\tau}$	-0.1428	0.9746	0.9600	0.9702	0.9590	0.9645	1.2792
	$r_{6,\tau}$	-0.0723	1.0465	0.9869	0.9875	0.9853	0.9886	1.2918
200	$r_{2,\tau}$	0.1400	0.6594	0.5609	0.5600	0.5595	0.5617	0.9090
	$r_{4,\tau}$	-0.1113	0.9599	0.9607	0.9580	0.9577	0.9604	1.1518
	$r_{6,\tau}$	-0.0360	0.9732	0.9885	0.9875	0.9875	0.9890	1.1731
500	$r_{2,\tau}$	0.0884	0.6086	0.5672	0.5668	0.5663	0.5672	0.7396
	$r_{4,\tau}$	-0.0531	0.9388	0.9579	0.9575	0.9576	0.9579	1.0610
	$r_{6,\tau}$	0.0216	0.9986	0.9910	0.9903	0.9917	0.9909	1.0925

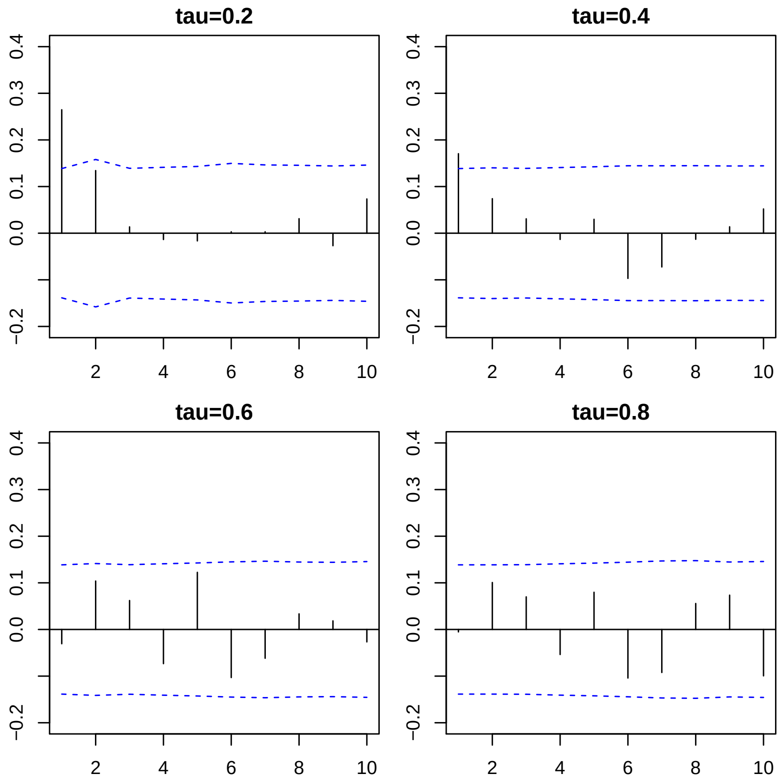

Figure 1: The sample QPACF of the observed time series, $\widetilde{\phi}_{kk,\tau}$, with $\tau = 0.2$, 0.4, 0.6, and 0.8. The dashed lines correspond to $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$.

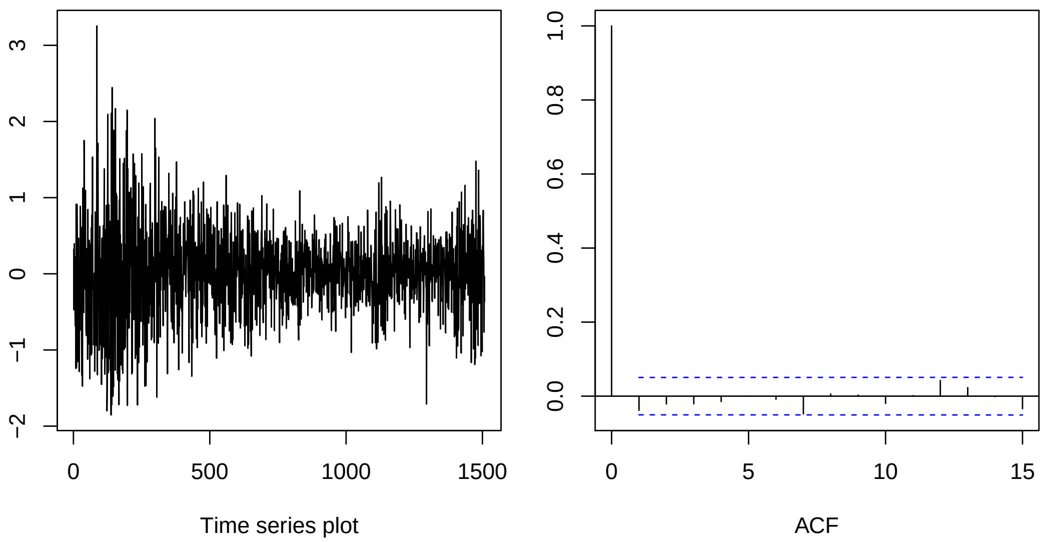

Figure 2: The time series plot and the sample ACF of the log return (as a percentage) of the daily closing price on the Nasdaq Composite from January 1, 2002 to December 31, 2007.

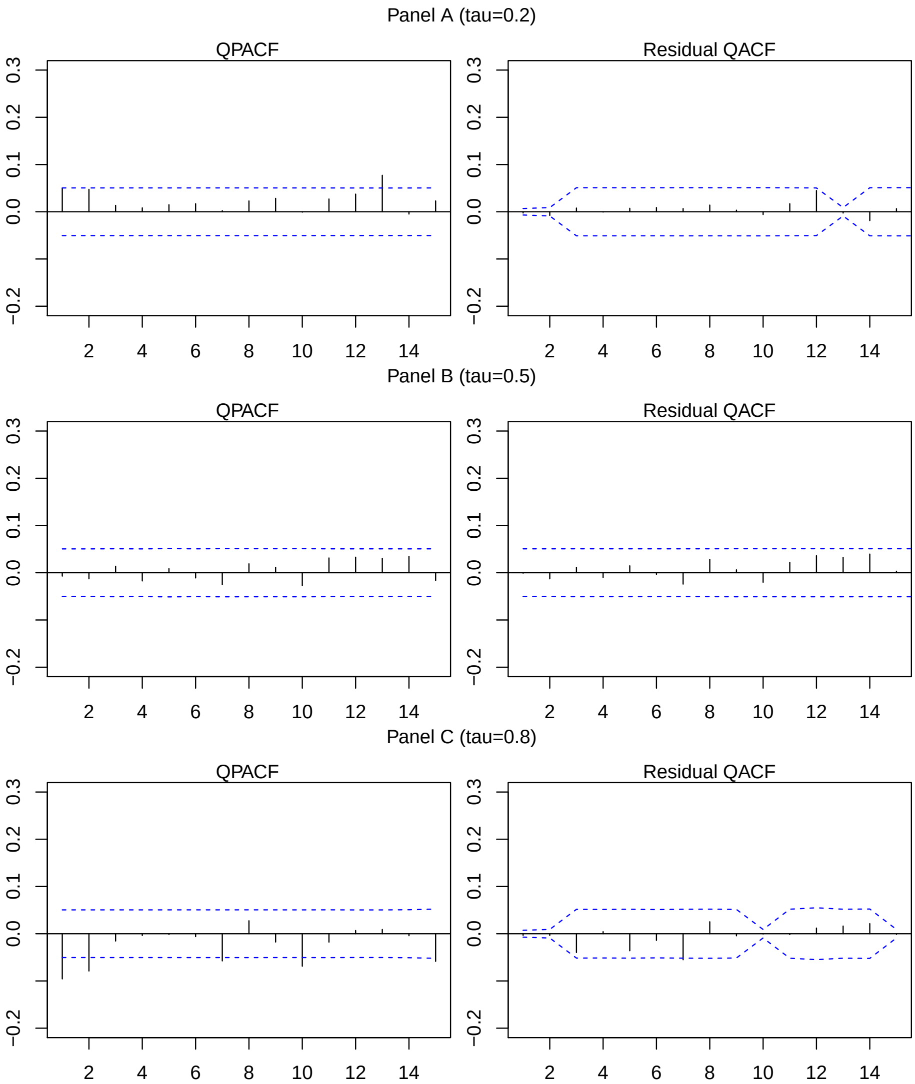

Figure 3: The sample QPACF of daily closing prices on the Nasdaq Composite and the sample QACF of residuals from the fitted models for $\tau = 0.2$, 0.5, and 0.8. The dashed lines in the left and right panels correspond to $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$ and $\pm 1.96\sqrt{\widehat{\Omega}_5/n}$, respectively.

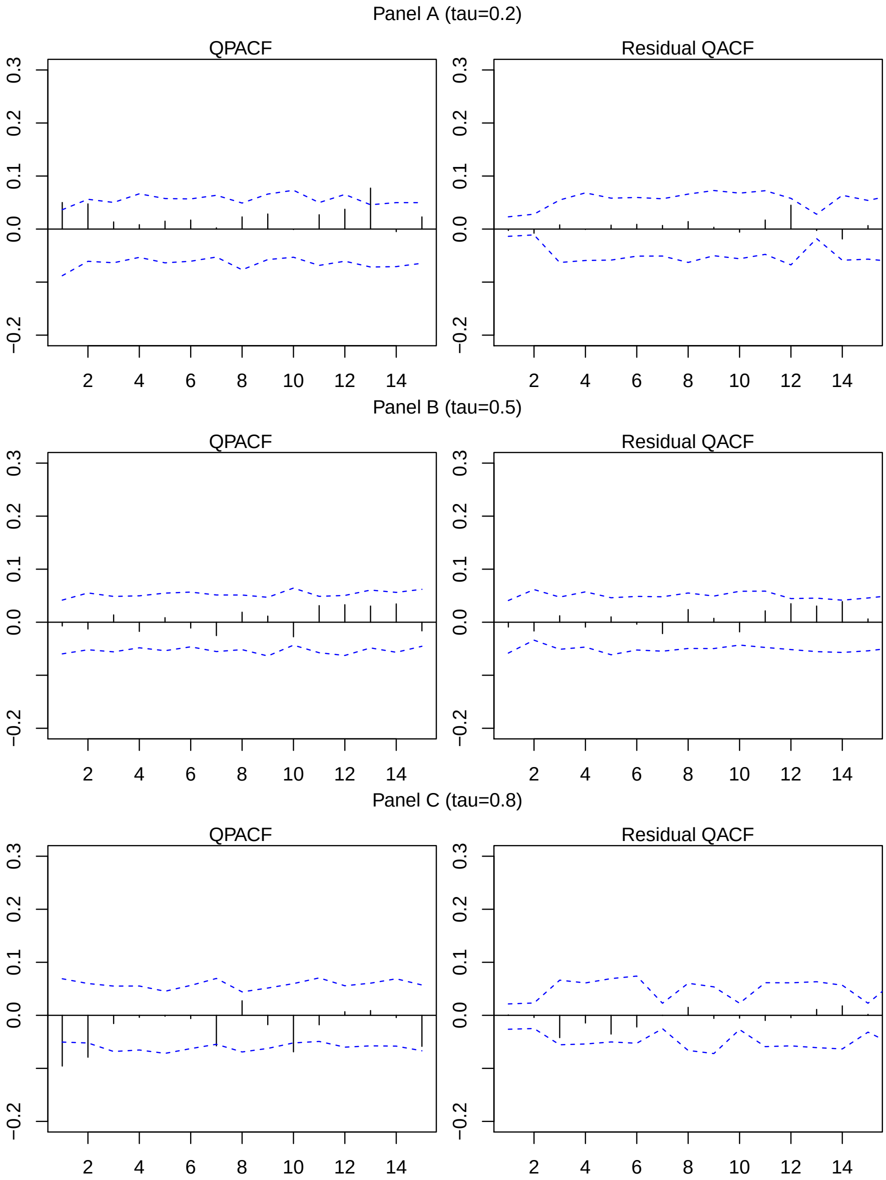

Figure 4: The sample QPACF of daily closing prices on the Nasdaq Composite and the sample QACF of residuals from the fitted models for $\tau = 0.2$, 0.5, and 0.8. The dashed lines in the left and right panels correspond to 2.5th and 97.5th percentiles of the bootstrapped distributions.