Optimal Sample Size Allocation for Accelerated Degradation Test

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Abstracts

Accelerated Degradation tests (ADTs) are widely used to assess the lifetime information of highly reliable products possessing quality characteristics that both degrade over time and can be related to reliability. Hence, how to design an efficient ADT plan for assessing product’s lifetime information at normal-use stress (especially for the optimal sample-size allocation to higher test-stress levels) turns out to be a challenging issue for reliability analysts. In the literature, several papers had addressed this decision problem. However, the results are only based on a specific degradation model (such as Wiener, Gamma, inverse Gaussian models, etc.) and it lacks of a uniform approach for a general degradation model. To overcome this difficulty, we first propose an exponential dispersion (ED) degradation model which covers all mentioned above degradation models. Next, by using V-optimality, D-optimality, and A-optimality criterion, we derive the analytical expression of the optimal sample-size allocation for a 2-stress ADT when the underlying degradation model follows an ED degradation model. The results demonstrate that the V-optimal and A-optimal allocations are the functions of unknown parameters and life-stress function, while D-optimal allocation turns out to be an equal sample-size allocation.

Keywords: Exponential dispersion model, V-optimality, D-optimality, A-optimality

1. Introduction

Assessing the reliability information of products is an essential task of enhancing continuously the product's quality and reliability. For highly reliable products, it seems difficult to assess the lifetime of the products by using traditional life tests that record only time-to-failure. Even the technique of accelerating the product by testing at higher levels of stress, such as elevated temperatures or voltages, is little help, since no failures are likely to occur over a reasonable period of time. In such conditions, if there exists a quality characteristic (QC) whose degradation over time can be related to reliability, we could obtain more information about the product's reliability by collecting the degradation data. For some highly reliable products, degradation path may degrade very slowly and it is therefore impossible to obtain a good estimate within a reasonable test time. To overcome this problem, the reliability information at the normal use condition can be extrapolated by collecting degradation data under higher stress levels. This is called an accelerated degradation test (ADT).

However, how to design an efficient ADT plan for assessing product’s lifetime information at normal-use stress (especially for the optimal sample-size allocation to higher test-stress levels) turns out to be a challenging issue for reliability analysis.

Basically, there are various forms of optimality criteria that can be used to address the optimal sample size allocation problem. Typically, D-optimality is useful tool for obtaining more precise estimations of model parameters. The concept of A-optimality, which also aims to estimate model parameters well, deals with the individual variances of the estimated model parameters. Finally, the concept of V-optimality considers the prediction variance over a specified region of product's reliability information (such as product's *q*-th percentile lifetime at use condition). Under these three criteria, Tseng *et al.* (2011) and Lim *et al.* (2011) discuss the problem for an ADT plane based on the Wiener process degradation models under the above 3 criteria. However, there is no literature that proposed a uniform approach for a general degradation model such as the gamma process and the inverse Gaussian process to design an efficient ADT plan.

In this paper, we first propose an exponential dispersion degradation model to describe the degradation paths of an ADT. Then by using the D-optimality, A-optimality, and V-optimality criterion, we can derive analytical results of the optimal sample size allocation formula for 2-stress ADT design problems.

2. Exponential Dispersion Degradation Model

A stochastic process $\{Y(t) \mid t>0\}$ is called an exponential dispersion (ED) degradation model if its independent increment $\Delta Y_j = Y(t_j) - Y(t_{j-1})$ has the following probability density function (pdf):

$$f(\Delta y_j \mid \theta, \lambda) = c(\Delta y_j;\ \lambda)\exp\left\{\lambda\left(\Delta y_j\theta - \Delta t_j\kappa(\theta)\right)\right\}, \tag{1}$$

where $\Delta t_j = t_j - t_{j-1}$, for all $j = 1, 2, ...,$ and θ and λ are the canonical and dispersion parameters, respectively. By setting $\mu = \kappa'(\theta) = \tau(\theta)$ and assuming exists, then Equation (1) can be expressed as follows:

$$f(\Delta y_j \mid \mu, \lambda) = c(\Delta y_j;\ \lambda)\exp\left\{\lambda\left[\Delta y_j\tau^{-1}(\mu) - \Delta t_j\kappa(\tau^{-1}(\mu))\right]\right\}. \tag{2}$$

The ED degradation process is denoted by $Y(t) \sim ED(\mu t, \lambda)$. From Jørgensen (1997), it can be shown that $E(Y(t)) = \mu t,$ and $\mathrm{Var}(Y(t)) = V(\mu)t/\lambda,$ where $V(\mu) = \kappa''(\tau^{-1}(\mu)) = \kappa''(\theta)$, for all $t > 0$. Hence, $V(\mu)$ is also known as the variance function. An important class of ED model can be proposed by the following power function,

$$V(\mu) = \mu^d, d \in R. \tag{3}$$

3. Problem Formulation

Suppose that we have *N* units which are available for conducting an ADT and the test conditions are described as follows:

(1) Let S_0 denote the normal-use stress and there are *k* higher stress levels $\{S_l\}_{l=1}^{k}$ on the ADT, that is, $(S_0) < S_1 < S_2 < \cdots < S_k$.

(2) We assign n_l units to the stress level S_l, where $1 \le l \le k$, and $\sum_{l=1}^{k} n_l = N$. Furthermore, let $p_l = (n_l / N)$ denote the proportion of sample size allocation to the *l*-th stress, $1 \le l \le k$.

(3) For each stress level, the degradation amounts of each test unit are collected at times $\{t_j\}_{j=1}^{m}$ and let $Y_i(t_j \mid S_l)$ denote the degradation path of the *i*-th product under stress S_l at time t_j. Assume that for all $i = 1,\dots,n_l$, $j = 1,\dots,m$, $l = 1,\dots,k$.

(4) For all $l = 1,\dots,k$, assume that there exists a suitable function $X(\bullet)$ such that

$$\ln(\mu(S_l)) = b_0' + b_1' X(S_l). \qquad \textbf{(4)}$$

Two familiar examples for are as follows (Nelson, 1990):

$$X(S_l) = \begin{cases} 1/S_l\ , \text{ Arrhenius model,} \\ \ln S_l\ , \text{ Inverse-power model.} \end{cases}$$

In the following, under the constraint of $\sum_{l=1}^{k} p_l = 1$, we will derive the optimal sample size allocation procedure by using the criteria of D-optimality, A-optimality, and V-optimality, respectively.

4. Main Results for *k*=2

Set $b_0 = b_0' + b_1' X(S_0)$, $b_1 = b_1'\left(X(S_k) - X(S_0)\right)$, and $x_l = \dfrac{X(S_l) - X(S_0)}{X(S_k) - X(S_0)}$.

Then, Equation (4) can be expressed as

$$\ln(\mu^*(x_l)) = b_0 + b_1 x_l .$$

Let $\Delta Y_{ijl} = Y_i(t_j \mid S_l) - Y_i(t_{j-1} \mid S_l)$ denote the increment of $Y_i(t_j \mid S_l)$, where $\Delta t_j = t_j - t_{j-1}$, $j = 1,\dots,m$, $l = 1,\dots,k$, and $i = 1,\dots n$. Then, the pdf of ΔY_{ijl} is

$$f(\Delta y_{ijl} \mid \mu^*(x_l), \lambda) = c(\Delta y_{ijl};\lambda)\exp\left\{\lambda\left[\tau^{-1}(\mu^*(x_l))\Delta y_{ijl} - \Delta t_j \kappa(\tau^{-1}(\mu^*(x_l)))\right]\right\},$$

and its corresponding log-likelihood function can be obtained as follows:

$$\begin{aligned}\ln L(\Theta) &= \prod_{l=1}^{k}\prod_{j=1}^{m}\prod_{i=1}^{n_l} f(\Delta y_{ijl} \mid \mu^*(x_l), \lambda) \\ &= \sum_{l=1}^{k}\sum_{j=1}^{m}\sum_{i=1}^{n_l}\left(\ln c(\Delta y_{ijl};\ \lambda) + \lambda\tau^{-1}(\mu^*(x_l))\Delta y_{ijl}\right) - N\lambda m\Delta t\sum_{l=1}^{k} p_l \kappa(\tau^{-1}(\mu^*(x_l))).\end{aligned}$$

Hence, the MLE for $\Theta = (b_0,\ b_1,\ \lambda)$, $\hat{\Theta} = (\hat{b}_0,\ \hat{b}_1,\ \hat{\lambda})$, can be obtained by maximizing the above log-likelihood function and using the best asymptotic normality (BAN) property, we have $\hat{\Theta} \sim N(\Theta, I^{-1}(\Theta))$, where the Fisher information matrix is given below:

$$I(\Theta) = Nm\lambda\Delta t \begin{bmatrix} \sum_{l=1}^{k} p_l a_{0l}^2 & \sum_{l=1}^{k} p_l a_{0l} a_{1l} & 0 \\ \sum_{l=1}^{k} p_l a_{0l} a_{1l} & \sum_{l=1}^{k} p_l a_{1l}^2 & 0 \\ 0 & 0 & \varsigma \end{bmatrix}.$$

where $a_{0l} = e^{\frac{-1}{2}(d-2)(b_0+b_1x_l)}$, $a_{1l} = x_l e^{\frac{-1}{2}(d-2)(b_0+b_1x_l)}$, $l = 1,\ldots,k,$ and

$$\varsigma = \frac{1}{\lambda\Delta t} E\left(-\frac{\partial^2 \ln c(\Delta y_{ijl};\ \lambda,\ \Delta t)}{\partial \lambda^2} \right).$$

In the D-optimality criterion, to obtain the optimal sample allocation is equivalent to maximization of the function

$$D(p_1,\ldots,p_k) = \sum_{i<j}^{k} e^{-b_1(d-2)\left(x_i+x_j\right)} (x_j - x_i)^2 p_i p_j .$$

In the A-optimality criterion, to obtain the optimal sample allocation is equivalent to minimize the function

$$A(p_1,\ldots,p_k) = \frac{\sum_{l=1}^{k} p_l e^{-b_1(d-2)x_l} (1+x_l^2)}{\sum_{i<j}^{k} e^{-b_1(d-2)\left(x_i+x_j\right)} (x_j - x_i)^2 p_i p_j}.$$

In the V-optimality criterion, by the delta-method, we have the following result:

$$\mathrm{AVar}(\hat{t}_q) = \frac{1}{Nf_0^2} \left(\frac{h_3^2}{\varsigma Nm\lambda\Delta t} + \frac{\sum_{l=1}^{k} p_l (h_2 a_{0l} - h_1 a_{1l})^2}{Nm\lambda\Delta t \left(\sum_{l=1}^{k} p_l a_{0l}^2 \sum_{l=1}^{k} p_l a_{1l}^2 - \left(\sum_{l=1}^{k} p_l a_{0l} a_{1l} \right)^2 \right)} \right), \quad \textbf{(5)}$$

where $h_3 = \dfrac{\partial F(\hat{t}_q;\ \mu^*(x_0),\ \lambda)}{\partial \lambda}$.

In Equation (5), note that h_3 is independent of $\{p_l\}_{l=1}^{k}$. Hence, in the V-optimality criterion, to minimize Equation (5) is equivalent to minimize

$$V\left(p_1, p_2, \cdots, p_k\right) = \mathbb{C}_0 \frac{\sum_{l=1}^{k} p_l e^{-b_1(d-2)x_l} x_l^2}{\sum_{i<j}^{k} e^{-b_1(d-2)\left(x_i+x_j\right)} \left(x_j - x_i\right)^2 p_i p_j}, \qquad \textbf{(6)}$$

where $\mathbb{C}_0$ is a constant which is independent of $\left(p_1, p_2, \cdots, p_k\right)$.

Under k=2, and since $p_2 = 1 - p_1$, we have the following results.

Theorem 1

When k=2, the optimal allocation subject to D-optimality criterion is

$$(p_1^*,\ p_2^*) = (0.5, 0.5).$$

Theorem 2

When *k*=2, the optimal allocation subject to A-optimality criterion is

$$(p_1^*,\, p_2^*) = \left(\frac{d}{1+d},\, \frac{1}{1+d} \right),$$

where $d = 2(1+x_1^2)^{-1} \exp\{-b_1(d-2)(1-x_1)/2\}$.

Theorem 3

When *k*=2, the optimal allocation subject to V-optimality criterion is

$$(p_1^*,\, p_2^*) = \left(\frac{r}{1+r},\, \frac{1}{1+r} \right),$$

where $r = x_1^{-1} \exp\{-b_1(d-2)(1-x_1)/2\}$.

5. An Illustrative Example

Carbon-film-resistors are widely used to reduce voltage and limit current in industry. In general, change in resistance can reduce product performance or even make system fail. However, the carbon-film-resistors degrade slowly at operating temperature. Thus, to assess the reliability of carbon film resistors at normal use condition $S_0 = 323.15$K (50 ℃), an ADT was performed from Park & Padgett (2006). A threshold value for percent increase in resistance is 20%, that is, $\xi = 1.2$. Carbon-film-resistors were observed at each hundred hour increment for 10 inspection times, and two acceleration levels for temperature are 350K and 450K. There are 10 test units for each stress level.

In Equation (3), if $d = 0, 2, 3$, then the corresponding degradation processes are Wiener, gamma and inverse Gaussian processes, respectively. Now, we use these three models to fit Carbon-film-resistors data (shown in Park & Padgett, 2006) and corresponding MLEs for unknown parameters $\Theta = (b_0, b_1, \lambda)$ are shown in Table 1.

Table 1. The ML estimators $\hat{\Theta}$

Degradation Model	*d*	$\hat{b}_0$	$\hat{b}_1$	$\hat{\lambda}$	AIC
Wiener process	0	1.19×10^{-4}	0.76	4.71×10^{5}	-1589.86
Gamma process	2	1.15×10^{-4}	0.80	1.97×10^{-2}	-1733.82
Inverse Gaussian	3	1.13×10^{-4}	0.83	1.87×10^{-6}	-1659.88

For illustrative purpose, it demonstrates that the gamma process has the smallest AIC among the above-mentioned degradation models. If we treat $\hat{\Theta}$ as the true parameters, then, the optimal sample size allocation can be obtained in Table 2.

Table 2. The optimal allocation under Gamma process

	p_1^*	p_2^*
D-optimal	0.50	0.50
A-optimal	0.65	0.35
V-optimal	0.79	0.21

From Table 2, in the gamma degradation model, if $N = 100$, then the optimal sample size allocation under the V-optimality is that we should allocate 79 units to the lower stress level (350K) and 21 units to the higher stress level (450K).

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