Hitting Time Distributions for Denumerable Birth and Death Processes

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Abstract

For an ergodic continuous-time birth and death process on the nonnegative integers, a well-known theorem states that the hitting time $T_{0,n}$ starting from state 0 to state n has the same distribution as n independent exponential random variables. Firstly, we generalize this theorem to absorbing birth and death process (say state -1 absorbing), to derive the distribution of $T_{0,n}$. We then give the explicit formulas for Laplace transforms of hitting times between any two states for an ergodic or absorbing birth and death process. Secondly these results are all extended to birth and death processes on the nonnegative integers with ∞ an exit, entrance, or regular boundary. Finally, we apply these formulas to fastest strong stationary times for strongly ergodic birth and death processes.

Keywords and phrases: Birth and death process, eigenvalues, hitting time, strong stationary time

1 Introduction

In this paper, we will study the distribution of passage time between any two states of an irreducible continuous-time birth and death process on the nonnegative integers $\{0, 1, 2, \cdots\}$. This is an extension of a well-known theorem, which states that the passage time from state 0 to state d $(< \infty)$ has the same law as a sum of d independent exponential random variables with distinct rates. These rates are just the non-zero eigenvalues of the associated generator for the process absorbed at state d. Cf.[3, 10, 12]. For historical comments, see Fill [7] and Diaconis and Miclo [4] and references therein.

Very recently, Fill [7] gave a first stochastic proof for this theorem via duality. An excellent application of this theorem is to the distribution of fastest strong stationary time for an ergodic birth and death process on $\{0, 1, \cdots, d\}$. And it is also the starting point of studying separation cut-off for birth and death processes in [5]. By a similar method, Fill [8] proved an analogous result for upward skip-free processes. Diaconis and Miclo [4] presented another probabilistic proof for birth and death processes, by using the "differential operators" for birth and death processes([6]).

Consider a continuous-time birth and death process $(X_t)_{t\geq 0}$ with generator $Q = (q_{ij})$ on $E = \{0, 1, 2, \cdots\}$. The (q_{ij}) are specified as follows:

$$q_{ij} = \begin{cases} b_i, & \text{for } j = i+1,\ i \geq 0; \\ a_i, & \text{for } j = i-1,\ i \geq 1; \\ -(a_i + b_i), & \text{for } j = i \geq 0; \\ 0, & \text{for other } j \neq i. \end{cases} \tag{1.1}$$

Here $\{a_i : i \geq 1\}$ and $\{b_i : i \geq 0\}$ are two sequences of positive numbers, and $a_0 \geq 0$. When $a_0 = 0$, state 0 is reflecting; when $a_0 > 0$, state 0 is not conservative, so by convention we can and do regard an extra state -1 as the absorbing state.

Let $T_{i,n}$ be the hitting time of state n starting from state i. A well known theorem is the following (cf.[7, Theorem 1.1]).

Theorem 1.1. *Let $\lambda_1^{(n)} < \cdots < \lambda_n^{(n)}$ be all (positive) n eigenvalues of $-Q^{(n)}$, where*

$$Q^{(n)} = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & -(a_2+b_2) & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & -(a_{n-1}+b_{n-1}) \end{pmatrix}. \tag{1.2}$$

Then $T_{0,n}$ is distributed as a sum of n independent exponential random variables with rate parameters $\left\{\lambda_1^{(n)}, \cdots, \lambda_n^{(n)}\right\}$. That is,

$$\mathbb{E}e^{-sT_{0,n}} = \prod_{\nu=1}^{n} \frac{\lambda_\nu^{(n)}}{s+\lambda_\nu^{(n)}}, \quad s \geq 0. \tag{1.3}$$

We will investigate the distribution of hitting time $T_{i,n}$ for birth and death process (X_t) on the nonnegative integers.

2 Finite state space with $a_0 = 0$

Let's first solve Case I from Theorem 1.1.

Corollary 2.1. *For $0 \leq i < n < \infty$,*

$$\mathbb{E}e^{-sT_{i,n}} = \frac{\displaystyle\prod_{\nu=1}^{n} \frac{\lambda_\nu^{(n)}}{s+\lambda_\nu^{(n)}}}{\displaystyle\prod_{\nu=1}^{i} \frac{\lambda_\nu^{(i)}}{s+\lambda_\nu^{(i)}}}, \qquad s \geq 0. \tag{2.1}$$

For $0 \leq n < N < \infty$, let $\widehat{\lambda}_{n,1}^{(N)} < \widehat{\lambda}_{n,2}^{(N)} < \cdots < \widehat{\lambda}_{n,N-n}^{(N)}$ be positive eigenvalues of $-\widehat{Q}_n^{(N)}$, where

$$\widehat{Q}_n^{(N)} := \begin{pmatrix} -(a_{n+1}+b_{n+1}) & b_{n+1} & 0 & 0 & \cdots & 0 & 0 \\ a_{n+2} & -(a_{n+2}+b_{n+2}) & b_{n+2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_N & -a_N \end{pmatrix}. \tag{2.2}$$

This is the generator of the birth and death process on $\{n+1, \cdots, N\}$ with reflecting state N before the process reaches state n, which can be considered an absorbing state.

By using mapping $j \mapsto j' = N - j$ on the state space, we can easily obtain the following results from Theorem 1.1 and Corollary 2.1.

Corollary 2.2. *For $0 \leq n < i \leq N < \infty$,*

$$\mathbb{E}e^{-sT_{i,n}} = \frac{\displaystyle\prod_{\nu=1}^{N-n} \frac{\widehat{\lambda}_{n,\nu}^{(N)}}{s+\widehat{\lambda}_{n,\nu}^{(N)}}}{\displaystyle\prod_{\nu=1}^{N-i} \frac{\widehat{\lambda}_{i,\nu}^{(N)}}{s+\widehat{\lambda}_{i,\nu}^{(N)}}}, \qquad s \geq 0.$$

3 Finite state space with $a_0 > 0$

In this section, we will study the hitting time distribution for absorbing finite space, and give the answers for Case I and Case II.

Now fix $n \geq 1$ and assume that $a_0 > 0$. Set

$$G = \begin{pmatrix} -(a_0+b_0) & b_0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & -(a_1+b_1) & b_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & -(a_{n-1}+b_{n-1}) & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

Let $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n$ be $n+1$ eigenvalues of $-G$, and denote by $P(t)$ the probability transition function corresponding to G. Note that states n and -1 are absorbing states for $P(t)$.

And let μ_i be defined as $\mu_0 = 1$ and $\mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}$ for $i \geq 1$,, and further define

$$h_i = \sum_{j=0}^{i} \frac{1}{\mu_i a_j}, \ (0 \leq i \leq n-1), \quad h_n = h_{n-1} + \frac{1}{\mu_{n-1} b_{n-1}}. \tag{3.2}$$

Theorem 3.1. *With notations given above, the hitting time $T_{0,n}$ has Laplace transform*

$$\mathbb{E}e^{-sT_{0,n}} = \frac{h_0}{h_n} \cdot \prod_{\nu=1}^{n} \frac{\lambda_\nu}{s+\lambda_\nu}, \quad s > 0.$$

4 Case III: ∞ is an exit boundary

Assume that $a_0 = 0$. When $R < \infty$, the corresponding Q-processes are not unique, for details see [1, Chapter 8] or [2, Chapter 4]. Let $(X_t, t \geq 0)$ be the corresponding continuous-time Markov chain with minimal Q-function $P(t) = (p_{ij}(t))$ on E, that is,

$$p_{ij}(t) = \mathbb{P}_i[X_t = j, \ t < \zeta]$$

with ζ the lifetime.

Denote by $L^2(\mu)$ the usual (real) Hilbert space on E. Then it is well known that $Q^{(n)}, Q, P(t)$ are self-adjoint operators on $L^2(\mu)$. When $\sigma_{\text{ess}}(Q) = \emptyset$, denote by $\lambda_1 < \lambda_2 < \cdots$ all the eigenvalues of $-Q$.

Theorem 4.1. *Assume ∞ is an exit boundary. Let ζ be the lifetime for the minimal process. Then*

$$\mathbb{E}_0 e^{-s\zeta} = \prod_{\nu=1}^{\infty} \frac{\lambda_\nu}{s+\lambda_\nu}, \quad s \geq 0.$$

That is,

$$\zeta \overset{\mathscr{L}}{=} \sum_{\nu=1}^{\infty} Y_\nu,$$

where $Y_\nu \sim \text{Exp}(\lambda_\nu)$ (for $1 \leq \nu < \infty$) are independent. And for any $i \geq 0$, let $T_{i,\infty} = \lim_{n\to\infty} T_{i,n}$. Then

$$\mathbb{E}_i e^{-s\zeta} = \mathbb{E}e^{-sT_{i,\infty}} = \frac{\prod_{\nu=1}^{\infty} \frac{\lambda_\nu}{s+\lambda_\nu}}{\prod_{\nu=1}^{i} \frac{\lambda_\nu^{(i)}}{s+\lambda_\nu^{(i)}}}, \quad s \geq 0.$$

5 Case IV: ∞ is an entrance boundary

In this section we will deal with Case IV for a birth and death process with ∞ an entrance boundary; i.e., $R = \infty, S < \infty$, and the corresponding Q-process is unique.

Since $S < \infty$ and $S \geq \frac{1}{\mu_0 b_0} \sum_{j\geq 1} \mu_j$, we have $\mu = \sum_{j\geq 0} \mu_j < \infty$. Let $\pi_i = \mu_i/\mu$. Then $\pi := (\pi_i, i \geq 0)$ is a probability measure on E, and the process is reversible with respect to π. Now we will consider the spectral theory for operators on the Hilbert space $L^2(\pi)$.

For $n \geq 0$, let

$$\widehat{Q}_n = \begin{pmatrix} -(a_{n+1} + b_{n+1}) & b_{n+1} & 0 & \cdots & \cdots & \cdots \\ a_{n+2} & -(a_{n+2} + b_{n+2}) & b_{n+2} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{5.1}$$

be the generator of the birth and death process on $\{n+1, n+2, \cdots\}$ before the process reaches state n.

Let $\widehat{\pi}^{(n)} := (\pi_i : i > n)$ and $\widehat{E}_n := \{n+1, n+2, \cdots\}$. It is easy to check that $\widehat{Q}_n$ is symmetric with respect to $\widehat{\pi}(n)$ and then $\widehat{Q}_n$ is a self-adjoint operator in $L^2(\widehat{E}_n, \widehat{\pi}^{(n)})$. When $\sigma_{\text{ess}}(\widehat{Q}_n) = \emptyset$, denote by $\widehat{\lambda}_{n,1} < \widehat{\lambda}_{n,2} < \cdots$ all the positive eigenvalues of $-\widehat{Q}_n$.

Theorem 5.1. *Assume the birth and death process is such that ∞ is an entrance boundary. (a) For $n \geq 1$, we have*

$$\mathbb{E}e^{-sT_{n,0}} = \frac{\prod_{\nu=1}^{\infty} \frac{\widehat{\lambda}_\nu}{s + \widehat{\lambda}_\nu}}{\prod_{\nu=1}^{\infty} \frac{\widehat{\lambda}_{n,\nu}}{s + \widehat{\lambda}_{n,\nu}}}, \quad s \geq 0. \tag{5.2}$$

(b) Let $T_{\infty,0} = \lim_{n\to\infty} T_{n,0}$. Then

$$\mathbb{E}e^{-sT_{\infty,0}} = \prod_{\nu=1}^{\infty} \frac{\widehat{\lambda}_\nu}{s + \widehat{\lambda}_\nu}, \quad s \geq 0. \tag{5.3}$$

6 Application to the fastest strong stationary time

We give the distribution of fastest strong stationary times.

Theorem 6.1. *For a strongly ergodic birth and death process (i.e., $R = \infty, S < \infty$) started at state 0, any fastest SST τ has distribution*

$$\mathbb{E}e^{-s\tau} = \prod_{\nu=1}^{\infty} \frac{\lambda_\nu}{s + \lambda_\nu}, \quad s \geq 0,$$

where $\{\lambda_\nu : \nu \geq 1\}$ are the positive eigenvalues of $-Q$, and Q is the generator in (1.1).

7 ∞ is a regular boundary

The minimal transition function behaves like a process with exit boundary, so we will follow the notations in Section 4. For simple, we will only present the distribution of lifetime for the minimal process when $a_0 = 0$.

Assume that $a_0 = 0$. Let ζ be the lifetime of the minimal process $P^{\min}(t)$ with

$$p_{ij}^{\min}(t) = \mathbb{P}_i[X_t = j, t < \zeta], \quad t \geq 0, \quad i, j \in E$$

Theorem 7.1. *Assume that ∞ is a regular boundary, i.e., $R < \infty, S < \infty$. Let ζ be the lifetime for the minimal process. Then*

$$\mathbb{E}_0 e^{-s\zeta} = \prod_{\nu=1}^{\infty} \frac{\alpha_\nu}{s+\alpha_\nu}, \quad s \geq 0,$$

where $\alpha_\nu, \nu \geq 1$ are the eigenvalues associated to the minimal process.

We now turn to the "maximal" process. In contrast to the minimal case, the "maximal" process is much more complicated. There are two ways to construct the "maximal" process: one is the probabilistic method in [14, Chapter 13], another is via Dirichlet form theory in [9], see also [2, Proposition 6.56]. Here we will use the the construction in [13]. We prove that the "maximal" process is just the limit process of a series of reflecting birth and death processes on $\{0, 1, \cdots, n\}$ as n goes to infinity. Actually, we prove that the convergence is in operator norm in $L^2(\pi)$ for the Laplace transform for these processes. Thus we can use the standard perturbation theory for linear operators to derive the distribution for hitting times of the "maximal" process. For the argument, the reader is urged to refer [11, Chapter IV].

Theorem 7.2. *Assume the birth and death process is such that ∞ is a regular boundary. Let $T_{n,0}$ be the hitting time of state 0 starting from state n for the "maximal" process, and $\{\widehat{\lambda}_\nu\}$ (resp. $\{\widehat{\lambda}_{n,\nu}\}$) be eigenvalues of generators for the "maximal" process before hitting to state 0 (resp. state n). Then for $n \geq 1$, we have*

$$\mathbb{E}e^{-sT_{n,0}} = \frac{\prod_{\nu=1}^{\infty} \frac{\widehat{\lambda}_\nu}{s+\widehat{\lambda}_\nu}}{\prod_{\nu=1}^{\infty} \frac{\widehat{\lambda}_{n,\nu}}{s+\widehat{\lambda}_{n,\nu}}}, \quad s \geq 0. \tag{7.1}$$

References

[1] Anderson, W. Continuous–time Markov chains. New York, Springer–Verlag, 1991

[2] Chen, M.F. From Markov chains to non-equilibrium particle systems (Second edition). Singapore, Word Scientific, 2004

[3] Brown, M., Shao, Y.S. Identifying coefficients in the spectral representation for first passage time distributions. Probab. Eng. Inform. Sci. 1, 69-74 (1987)

[4] Diaconis, P., Miclo, L. On times to quasi-stationarity for birth and death processes. J. Theor. Probab., 22, 558-586 (2009)

[5] Diaconis, P., Saloff-Coste, L. Separation cut-offs for birth and death chains. Ann. Appl. Probab., 16, 2098-2122 (2006)

[6] Feller, W. The birth and death processes as diffusion processes. J. Math. Pures Appl., 38, 301-345 (1959)

[7] Fill, J.A. The passage time distribution for a birth-and-death chain: strong stationary duality gives a first stochastic proof. J. Theor. Probab., 22, 543-557 (2009)

[8] Fill, J.A. On hitting times and fastest strong stationary times for skip-free and more general chains. J. Theor. Probab., 22, 587-600 (2009)

[9] Hou, Z.-T., Chen, M.-F. On the symmetrizability of a class of Q-processes (in Chinese), J. Beijing Normal Univ. 3-4, 1-12 (1980)

[10] Karlin, S., McGregor, J. Coincidence properties of birth and death processes. Pac. J. Math., 9, 1109-1140 (1959)

[11] Kato,T. Purturbation theory for linear operators. Springer-Verlag, 1966

[12] Keilson, J. Markov chain models–rarity and exponentiality. Applied Mathematical Sciences, vol. 28. Springer, New York, 1979

[13] Wang Z.K., Yang X.Q. Birth and death processes and Markov processes. Science Press/Springer-Verlag, 1992

[14] Yang, X.-Q. The construction theory of denumerable Markov processes. John Wiley & Sons, 1990.