A Structural Approach to Credit Risk in a Markov Modulated Market

Tamal Banerjee, Mrinal K Ghosh and Srikanth K Iyer

Indian Institute of Science, Bangalore, India

Corresponding author: Mrinal K Ghosh, email: mkg@math.iisc.ernet.in

Abstract

We address the problem of pricing a defaultable bond in a Markov modulated market. Using Merton's structural approach we show that a defaultable bond is a European type contingent claim. Since the market is incomplete we use the method of quadratic hedging and minimal martingale measure to derive locally risk minimizing derivative price. The price of defaultable bonds is obtained as a solution to a system of PDEs with weak coupling subject to appropriate terminal condition.

Keywords: Structural approach, defaultable bond, minimal martingale measure, quadratic hedging.

1 Introduction and Model Description

We address the problem of pricing defaultable bonds using Merton's structural approach [5] in a Markov modulated market. In this approach explicit assumptions are made about the dynamics of a firm's assets, its capital structure, its debt and share holders. A firm defaults when its value reaches a certain lower threshold, defined *endogenously* within the model. In this approach, a corporate liability can be characterized as an option on the firm's assets.

Let $(\Omega, \mathcal{F}, P)$ be the underlying complete probability space. We assume that the market can only be in a finite number of states. We model the market dynamics by a continuous time irreducible Markov chain $X = \{X_t\}_{t \geq 0}$, on a finite state space $\mathcal{X} = \{1, 2, \ldots, k\}$ with Q-matrix $\Lambda = [\lambda_{ij}]$. Define the following functions

$$r : \mathcal{X} \mapsto (0, \infty), \ \mu : \mathcal{X} \mapsto (0, \infty), \ \kappa : \mathcal{X} \mapsto \mathbb{R}, \ \sigma : \mathcal{X} \mapsto (0, \infty). \tag{1.1}$$

We assume that the floating interest rate $r(X_t)$, the drift coefficient $\mu(X_t)$, the cash payout rate $\kappa(X_t)$ and the volatility $\sigma(X_t)$ depend only on the current state X_t of the market. The asset value of the firm A_t is assumed to follow a Markov modulated geometric Brownian motion given by

$$dA_t = A_t \left[\left(\mu(X_t) - \kappa(X_t) \right) dt + \sigma(X_t) \, dW_t \right], \quad A_0 > 0, \tag{1.2}$$

where $\{W_t\}_{t \geq 0}$ is a standard Wiener process independent of $X = \{X_t\}_{t \geq 0}$.

Let $\{B_t\}_{t \geq 0}$ be the amount in a money market account at time t with floating interest rate $r(X_t)$ given by

$$B_t = e^{\int_0^t r(X_u) du}. \tag{1.3}$$

Let $\mathcal{F}_t = \sigma(A_u, X_u, \ u \leq t)$. Without loss of generality we assume that $\{\mathcal{F}_t\}$ is right-continuous and complete. This would be the basic filtration for our model. In the structural approach it is assumed that the company has a simple capital structure consisting of one debt obligation (or defaultable bond) and one type of equity. The debt and equity may be viewed as contingent claims on the firm's assets. Let $\{E_t\}_{t \geq 0}$ denote the equity process of the company which is traded publicly. Suppose the process $\{D_t\}_{t \geq 0}$ denotes the market value of the defaultable bond of the company. The company defaults if $A_T < K$, where T is the maturity time and K the face value of the defaultable bond. If the company defaults, then the payoff to the equity holders is zero. If it does not, i.e., $A_T \geq K$, then there is a net profit of $A_T - K$ after paying back the debt. Thus the total payoff to equity holders at maturity is

$$E(T, A_T, X_T) = (A_T - K)^+ := \max(A_T - K, 0), \tag{1.4}$$

which is identical to the payoff for a European call option on A_t with strike price K, dividend rate $\kappa(X_t)$ and maturity T. Therefore for $t \in [0, T]$, the equity process is a long European call from the point of view of the equity holders. The value of the defaultable bond at time T is given by

$$D(T, A_T, X_T) = \min(K, A_T) = K - (K - A_T)^+. \tag{1.5}$$

The above payoff is equivalent to that of a portfolio consisting of a default free loan with face value K maturing at T and a short European put option on A_t with dividend rate $\kappa(X_t)$, strike price K and maturity T. Since P is the market measure, this approach allows for estimation of model parameters from market data without recourse to other bond prices. We now discuss pricing of such European type contingent claim as in [1]. Let H be a European type contingent claim maturing at time T $(T > 0)$. In order to determine the price of this contingent claim at any time $0 \leq t < T$, we first find an equivalent martingale measure (EMM) P^* for this model. Let the discounted asset price be A_t^*, where $A_t^* := B_t^{-1} A_t$, A_t and B_t being given by (1.2), (1.3). It is easy to see that A_t^* satisfies

$$dA_t^* = \left(\mu(X_t) - r(X_t) - \kappa(X_t)\right) A_t^* dt + \sigma(X_t) A_t^* dW_t. \tag{1.6}$$

For $i \in \mathcal{X}$, define the *market price of risk* in regime i as $\gamma(i) := \frac{\mu(i) - r(i)}{\sigma(i)}$. Therefore, in each regime i, the quantity $\gamma(i)$ gives the excess return on the asset over the money market account per unit of volatility. Let

$$\rho_t := \exp\left(-\int_0^t \gamma(X_u) dW_u - \frac{1}{2} \int_0^t \gamma(X_u)^2 du\right), \ 0 \leq t \leq T, \tag{1.7}$$

and

$$dP^* := \rho_T \ dP. \tag{1.8}$$

Note that $\{\rho_t\}_{0 \leq t \leq T}$ is a square integrable P-martingale and P^*, as defined above is a probability measure equivalent to P. Under P^*

$$W_t^* := W_t + \int_0^t \gamma(X_u) du. \tag{1.9}$$

is an $\{\mathcal{F}_t\}$-Wiener process, and the asset process A_t satisfies

$$dA_t = A_t \left[\left(r(X_t) - \kappa(X_t)\right) dt + \sigma(X_t) dW_t^*\right].$$

Also under P^*, the discounted asset price satisfies

$$dA_t^* \; = \; A_t^*\Big[-\kappa(X_t)dt + \sigma(X_t)dW_t^*\Big]. \tag{1.10}$$

Since $\left\{A_t^* + \int_0^t A_u^*\kappa(X_u)du\right\}$ is a martingale under P^*, an *equivalent martingale measure* (EMM) for this model is given by P^*. This ensures that the market is arbitrage free. Let E^* denote the expectation under P^*. Therefore an arbitrage free price for an attainable contingent claim H at time $0 \leq t \leq T$ is given by,

$$B_t E^*[B_T^{-1} H \mid \mathcal{F}_t] = E^*[e^{-\int_t^T r(X_u)du} H \mid \mathcal{F}_t]. \tag{1.11}$$

Note that though the pricing formula is the same for both dividend and non-dividend paying assets, the evolution of the asset A_t is different under the measure P^*. Under P^*, the discounted asset price is no longer a martingale for dividend paying assets. It is well known that for a complete market, the EMM, P^* is unique. However our model is not complete due to the additional uncertainty described by the regime switching. Therefore the EMM, P^* is not unique. As a result the price of the claim H in (1.11) is also not unique. Moreover, in an incomplete market, the claim H may not be attainable by a self-financing strategy. Therefore there is an intrinsic risk associated with every contingent claim. The writer must thus take care of the risk associated with a hedging strategy. Thus one looks for a "risk minimizing" price which we describe in the next section.

2　Quadratic Hedging Approach

In this section we determine the price of the defaultable bond for using quadratic hedging [4]. We assume that the contingent claim H at time T satisfies $H \in L^2(\Omega, \mathcal{F}_T, P)$. For replicating this claim we consider a strategy $\pi = \{\pi_t\}_{0 \leq t \leq T} = \{\xi_t, \eta_t\}_{0 \leq t \leq T}$, where ξ_t and η_t denote the amounts invested in asset A_t and money market account B_t respectively at time t. The process $\xi = \{\xi_t\}_{0 \leq t \leq T}$ is assumed to be predictable satisfying appropriate integrability conditions and $\eta = \{\eta_t\}_{0 \leq t \leq T}$ is an adapted process satisfying a suitable integrability condition. The value of the portfolio at time t under the strategy π is given by: $V_t(\pi) := \xi_t A_t + \eta_t B_t$. Hence the discounted value of the portfolio becomes $V_t^*(\pi) = \xi_t A_t^* + \eta_t$. The accumulated additional cash flow up to time t (also called the *discounted cost process*) is defined by the following right continuous square integrable process $\tilde{C}_t(\pi)$

$$\tilde{C}_t(\pi) := V_t^*(\pi) - \int_0^t \xi_u dA_u^* - \int_0^t \xi_u \kappa(X_u) A_u^* du, \; 0 \leq t \leq T. \tag{2.1}$$

A strategy π is *self-financing* if $\tilde{C}_t(\pi)$ is a constant. A contingent claim is called *attainable* if there is a self-financing hedging strategy. Since the market under consideration is not complete, every contingent claim H may not be attainable. Hence, instead of a self-financing strategy we look for a strategy π which minimizes, at each time t, the *residual risk*, given by

$$R_t(\pi) := E\Big[(\tilde{C}_T(\pi) - \tilde{C}_t(\pi))^2 \mid \mathcal{F}_t\Big], \tag{2.2}$$

over all admissible hedging strategies. We say that a hedging strategy π^* is *risk-minimizing* if

$$0 \leq R_t(\pi^*) \leq R_t(\pi), \; 0 \leq t \leq T,$$

for any other admissible strategy π. As noted in [4], though the notion of risk minimizing hedging strategy is quite natural, it is technically difficult to work with when the market measure P is not itself a martingale measure. This motivates the introduction of a weaker notion where the risk is minimized locally . We have the following equivalent definition ([4]) for a locally risk minimizing strategy.

Definition 2.1. An admissible strategy π^* is said to be *optimal* (i.e., locally risk minimizing) if the corresponding discounted cost $\{\tilde{C}_t(\pi^*)\}$ as in (2.1) is a square integrable martingale orthogonal to $\{M_t\}$ where

$$M_t := \int_0^t \sigma(X_u) A_u^* dW_u. \tag{2.3}$$

Let $H^* := B_T^{-1} H$. The existence of an optimal strategy for hedging H is equivalent to ([4]) the existence of *Föllmer-Schweizer decomposition* of H^* in the form

$$H^* = H_0 + \int_0^T \xi_u^{H^*} \left(dA_u^* + \kappa(X_u) A_u^* du \right) + L_T^{H^*}, \tag{2.4}$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P)$, $\xi^{H^*} = \{\xi_t^{H^*}\}$ is predictable and $L^{H^*} = \{L_t^{H^*}\}_{0 \le t \le T}$ is a square integrable martingale orthogonal to $\{M_t\}$ (as in (2.3)). For the decomposition (2.4), the corresponding optimal strategy $\pi_t^* = (\xi_t^*, \eta_t^*)$ (Theorem 3.14, [4]) is given by

$$\xi_t^* := \xi_t^{H^*}, \ \eta_t^* := V_t^* - \xi_t^* A_t^*, \tag{2.5}$$

with

$$V_t^* := H_0 + \int_0^t \xi_u^{H^*} \left(dA_u^* + \kappa(X_u) A_u^* du \right) + L_t^{H^*}, \ 0 \le t \le T. \tag{2.6}$$

Thus, the discounted cost process $\tilde{C}_t(\pi^*)$, corresponding to the optimal strategy π^* is given by

$$\tilde{C}_t(\pi^*) = H_0 + L_t^{H^*}. \tag{2.7}$$

Note that the *Föllmer-Schweizer decomposition* obtained above is for dividend paying assets. The effect of the dividend process in the above decomposition is also implicit in the $\{L_t^{H^*}\}_{0 \le t \le T}$ term. If the discounted asset price with the dividend added would have been a martingale under P, then one could have used the *Kunita-Watanabe decomposition* for determining the defaultable bond prices. However, as noted earlier the discounted asset price with dividend added is a semi-martingale under P. Hence we would not be able to use the usual Kunita-Watanabe projection technique directly. Therefore following [4] we define the minimal martingale measure for our model.

Definition 2.2. An EMM P' equivalent to P is said to be *minimal* if $P' \equiv P$ on $\mathcal{F}_0$ and if any square integrable P- martingale which is orthogonal to $\{M_t\}$ under P remains a martingale under P'.

It can be shown as in [1] that the EMM P^* (as in 1.8) is the unique minimal martingale measure. By the discussions following Definition 3.2 in [3], we conclude that the locally risk minimizing price of the claim H is given by (1.11). Therefore, though the expression for locally risk minimizing price remains the same for both attainable and non attainable claims, the residual risk for a non attainable claim is strictly positive. Note that if we can obtain the *Föllmer-Schweizer decomposition*

for a given contingent claim as in (2.4), then we can determine the corresponding optimal strategy from (2.5). Let the contingent claim H be given by

$$H = K - (K - A_T)^+. \tag{2.8}$$

Note that (2.8) is the value of the defaultable bond at maturity in our model. Consider the function

$$\varphi(t, A_t, X_t) = E^* \left[e^{-\int_t^T r(X_u)du} \left(K - (K - A_T)^+ \right) \Big| \mathcal{F}_t \right]. \tag{2.9}$$

It is easy to see that the joint process (A_t, X_t) is Markov under the EMM P^*, with generator given by

$$\mathcal{L}g(s, i) = \left(r(i) - \kappa(i) \right) s \frac{\partial}{\partial s} g(s, i) + \frac{1}{2} \sigma^2(i) s^2 \frac{\partial^2}{\partial s^2} g(s, i) + \sum_{j \in \mathcal{X}} \lambda_{ij} g(s, j). \tag{2.10}$$

Therefore by Feynman-Kac formula, $\varphi(t, A_t, X_t)$ is a mild solution to the following system of PDEs for $t < T$ and $i \in \mathcal{X}$,

$$\frac{\partial \varphi(t, s, i)}{\partial t} + \frac{1}{2} \sigma(i)^2 s^2 \frac{\partial^2 \varphi(t, s, i)}{\partial s^2} + \left(r(i) - \kappa(i) \right) s \frac{\partial \varphi(t, s, i)}{\partial s} + \sum_{j=1}^k \lambda_{ij} \varphi(t, s, j) = r(i) \varphi(t, s, i), \tag{2.11}$$

with terminal condition

$$\varphi(T, s, i) = K - (K - s)^+, \ \forall \, i \in \mathcal{X}. \tag{2.12}$$

Thus the system of PDEs in (2.11) becomes the governing PDE for determining the defaultable bond price in our model. By solving the PDE (2.11) subject to terminal condition (2.12), we determine arbitrage free price of the defaultable bond in our model. To this end consider the system of PDEs given in (2.11). Define the set $\mathcal{A}$ as

$$\mathcal{A} = \left\{ f \, \Big| f \in C([0, T] \times \mathbb{R}) \bigcap C^{1,2}((0, T) \times \mathbb{R}), \ f \text{ has at most polynomial growth} \right\}. \tag{2.13}$$

The Cauchy problem (2.11)-(2.12) has a unique solution $\{\varphi^{(1)}(t, s, i), \ i = 1, 2, \ldots, k\}$ in the class of functions belonging to the set $\mathcal{A}$ ([1]). We obtain the following result by methods similar to those used in [1].

Theorem 2.1. *The unique solution $\{\varphi^{(1)}(t, s, i), \ i = 1, 2, \ldots, k\}$ of the Cauchy problem (2.11)-(2.12) is the risk minimizing price of the defaultable bond at time t, i.e., $\varphi^{(1)}(t, A_t, X_t) = \varphi(t, A_t, X_t)$.*

Proof. Let $0 \leq t \leq T$. By applying Ito's formula to $e^{-\int_0^t r(X_u)du} \varphi^{(1)}(t, A_t, X_t)$ under the measure P and using (1.6) and (2.11), we obtain after suitable rearrangement of terms

$$e^{-\int_0^t r(X_u)du} \varphi^{(1)}(t, A_t, X_t) = \varphi^{(1)}(0, A_0, X_0) + \int_0^t \frac{\partial \varphi^{(1)}(u, A_u, X_{u-})}{\partial s} \left(dA_u^* + \kappa(X_u) A_u^* du \right)$$

$$+ \int_0^t e^{-\int_0^u r(X_v)dv} \int_{\mathbb{R}} [\varphi^{(1)}(u, A_u, X_{u-} + h(X_{u-}, z)) - \varphi^{(1)}(u, A_u, X_{u-})] \hat{\wp}(du, dz),$$

$$\tag{2.14}$$

where $\hat{\wp}$ is a suitable centered Poisson random measure associated with the Markov chain $\{X_t\}$. Letting $t \uparrow T$, we obtain

$$B_T^{-1}(K - (K - A_T)^+) = \varphi^{(1)}(0, A_0, X_0) + \int_0^T \frac{\partial \varphi^{(1)}(u, A_u, X_{u-})}{\partial s} \left(dA_u^* + \kappa(X_u) A_u^* du \right)$$

$$+ \int_0^T e^{-\int_0^u r(X_v) dv} \int_{\mathbb{R}} [\varphi^{(1)}(u, A_u, X_{u-} + h(X_{u-}, z)) - \varphi^{(1)}(u, A_u, X_{u-})] \hat{\wp}(du, dz).$$

$$(2.15)$$

Clearly, the last term on the right in (2.14) is a P martingale orthogonal to the martingale $\{M_t\}$ (as defined in (2.3)). As remarked earlier, since P^* is the unique minimal martingale measure, the locally risk minimizing price will be given by (1.11). From (1.10) and the definition of a minimal martingale measure, it follows that the process in (2.14) is a P^* martingale. Combining these two facts we get the desired result.

$$\square$$

Remarks 2.1. *In this paper we have considered the case when default occurs only at time T. One can also study the situation when the default occurs when the asset value touches a lower threshold [2]. Using Föllmer-Schweizer decomposition we can also get explicit expressions for hedging strategies and residual risk; see [2] for more details. Using extensive numerical investigation it has been shown in [2] that the credit spread in Markov modulated case is higher than that in Merton's model.*

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